18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Properties of T (Lecture 19)

Let V be a finite dimensional vector space over $\mathbf{F}_2$. In this lecture, we will continue to establish some of the basic properties of Lannes' T-functor T_V. More precisely, we will show that T_V commutes with the functor Φ and with the formation of tensor products.

To begin, we observe that for every unstable $\mathcal{A}$-module M comes equipped with a canonical map

$$M \to (T_V M) \otimes \mathrm{H}^*(BV).$$

This induces a map

$$\Phi M \to \Phi(T_V M \otimes \mathrm{H}^*(BV)) \simeq (\Phi T_V M) \otimes (\Phi\, \mathrm{H}^*(BV)).$$

Composing with the Frobenius map $\Phi\, \mathrm{H}^*(BV) \to \mathrm{H}^*(BV)$, we obtain a map

$$\Phi M \to (\Phi T_V M) \otimes \mathrm{H}^* M,$$

which is adjoint to a map

$$h_M : T_V \Phi M \to \Phi T_V M.$$

Proposition 1. *For every unstable $\mathcal{A}$-module M, the map $h_M : T_V \Phi M \to \Phi T_V M$ is an isomorphism.*

Proof. Choose a resolution

$$\oplus_\beta F(n_\beta) \to \oplus_\alpha F(n_\alpha) \to M \to 0.$$

Since the functors T_V and Φ both preserve cokernels and direct sums, we conclude that h_M is an isomorphism provided that the maps $h_{F(n)}$ are isomorphisms, for each $n \geq 0$. We now work by induction on n, the case $n = 0$ being obvious.

Recall that we have an exact sequence

$$0 \to \Phi F(n) \to F(n) \to \Sigma\Omega F(n) \to 0.$$

Applying T_V, we obtain another exact sequence

$$0 \to T_V \Phi F(n) \to T_V F(n) \to T_V \Sigma\Omega F(n) \to 0.$$

The functor T_V commutes with Σ and Ω, so we can identify $T_V \Phi F(n)$ with the kernel K of the unit map $T_V F(n) \to \Sigma\Omega T_V F(n)$. On the other hand, we have an exact sequence

$$\Phi T_V F(n) \to T_V F(n) \to \Sigma\Omega T_V F(n) \to 0,$$

which determines a surjective map $g : \Phi T_V F(n) \to K$. The module $T_V F(n)$ is a direct sum of free unstable $\mathcal{A}$-modules, and therefore reduced. It follows that g is also injective, and determines an isomorphism $\Phi T_V F(n) \simeq T_V \Phi F(n)$. We now observe that this map is an inverse to $h_{F(n)}$. □

We now discuss the behavior of T_V with tensor products. Let M and N be unstable $\mathcal{A}$-modules. We have unit maps

$$M \to T_V M \otimes \mathrm{H}^*(BV)$$
$$N \to T_V N \otimes \mathrm{H}^*(BV).$$

Tensoring these together and composing with the multiplication on $\mathrm{H}^*(BV)$, we get a map

$$M \otimes N \to T_V M \otimes T_V N \otimes \mathrm{H}^*(BV)$$

which has an adjoint

$$\mu_{M,N} : T_V(M \otimes N) \to T_V M \otimes T_V N.$$

Our goal is to prove the following:

Theorem 2. *For every pair of unstable $\mathcal{A}$-modules M and N, the map*

$$\mu_{M,N} : T_V(M \otimes N) \to T_V M \otimes T_V N$$

is an isomorphism.

The proof proceeds in a series of steps. We begin with the following observation:

Remark 3. Let $V = V_0 \oplus V_1$. Then we have a canonical isomorphism

$$\mathrm{H}^*(BV) \simeq \mathrm{H}^*(BV_0) \otimes \mathrm{H}^*(BV_1).$$

It follows that the functor $M \mapsto M \otimes \mathrm{H}^*(BV)$ can be written as a composition of functors, given by tensor product with $\mathrm{H}^*(BV_0)$ and $\mathrm{H}^*(BV_1)$ respectively. Passing to left adjoints, we get a canonical isomorphism

$$T_V \simeq T_{V_0} \circ T_{V_1}.$$

The isomorphism of Remark 3 is compatible with the construction of the transformations $\mu_{M,N}$. Consequently, to prove Theorem 2, it will suffice to treat the case where $V \simeq \mathbf{F}_2$ is one-dimensional.

Notation 4. If $V = \mathbf{F}_2$, then we denote Lannes' T-functor simply by T.

The following is a special case of Theorem 2:

Lemma 5. *For every unstable $\mathcal{A}$-module N, the canonical map*

$$T(F(1) \otimes N) \to T(F(1)) \otimes T(N)$$

is an isomorphism.

Let us assume Lemma 5 for the moment, and use it to complete the proof of Theorem 2 in general.

Proof of Theorem 2. We wish to show that a canonical map

$$T(M \otimes N) \to T(M) \otimes T(N)$$

is an isomorphism. As functors of M, both sides are compatible with the formation of cokernels and direct sums. We may therefore argue as in the proof of Proposition 1 to reduce to the case where $M \simeq F(m)$ is a free module. Recall that $F(m)$ is canonically isomorphic to Σ_m-invariants in the tensor product $F(1)^{\otimes m}$. Since the functor T is exact, it commutes with the formation of fixed points. It will therefore suffice to prove the result in the case $M = F(1)^{\otimes m}$. We have a commutative diagram

$$\begin{array}{ccc} T(F(1)^{\otimes m} \otimes N) & \xrightarrow{\mu_{M,N}} & T(F(1)^{\otimes m}) \otimes T(N) \\ & \searrow^{\mu'} & \downarrow{\scriptstyle \mu'} \\ & & T(F(1))^{\otimes m} \otimes T(N). \end{array}$$

It follows from repeated application of Lemma 5 that the maps μ' and μ'' are isomorphisms, so that $\mu_{M,N}$ is an isomorphism as well. $\square$

Proof of Lemma 5. We wish to show that the canonical map

$$\mu_{F(1),N} : T(F(1) \otimes N) \to T(F(1)) \otimes T(N)$$

is an isomorphism. As functors of N, both sides preserve direct sums and cokernels. We may therefore assume that $N \simeq F(n)$ is a free unstable $\mathcal{A}$-module. We proceed by induction on n. We need to prove three things:

(*a*) The map $\mu_{F(1),N}$ is an isomorphism in every positive degree k. To prove this, we observe that N is reduced, so we have a map of exact sequences

$$\begin{array}{ccccccc} 0 \to T(F(1) \otimes \Phi N) & \longrightarrow & T(F(1) \otimes N) & \longrightarrow & T(F(1) \otimes \Sigma F(n-1)) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mu_{F(1),\Phi N}} & & \downarrow{\scriptstyle \mu_{F(1),N}} & & \downarrow{\scriptstyle \mu_{F(1),\Sigma F(n-1)}} & & \\ 0 \to T(F(1)) \otimes T(\Phi(N)) & \longrightarrow & T(F(1)) \otimes T(N) & \longrightarrow & T(F(1)) \otimes T(\Sigma F(n-1)) & \longrightarrow & 0. \end{array}$$

Since T commutes with suspension, the inductive hypothesis guarantees that $\mu_{F(1),\Sigma F(n-1)}$ is an isomorphism. Consequently, to show that $\mu_{F(1),N}$ is an isomorphism in degree k, it will suffice to show that $\mu_{F(1),\Phi N}$ is an isomorphism in degree k. We have a second map of exact sequences

$$\begin{array}{ccccccc} 0 \to T(\Phi F(1) \otimes \Phi N) & \longrightarrow & T(F(1) \otimes \Phi N) & \longrightarrow & T(\Sigma F(0) \otimes \Phi N) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mu_{\Phi F(1),\Phi N}} & & \downarrow{\scriptstyle \mu_{F(1),\Phi N}} & & \downarrow{\scriptstyle \mu_{\Sigma F(0),\Phi N}} & & \\ 0 \to T(\Phi F(1)) \otimes T(\Phi(N)) & \longrightarrow & T(F(1)) \otimes T(\Phi N) & \longrightarrow & T(\Sigma F(0)) \otimes T(\Phi N) & \longrightarrow & 0. \end{array}$$

Since T commutes with Σ, the map $\mu_{\Sigma F(0),\Phi F(n)}$ is an isomorphism. Consequently, to prove that $\mu_{F(1),N}$ is an isomorphism in degree k, it will suffice to show that $\mu_{\Phi F(1),\Phi N}$ is an isomorphism in degree k. Since T commutes with Φ, this is equivalent to the assertion that $\mu_{F(1),N}$ is an isomorphism in degree $\frac{k}{2}$, which follows from the inductive hypothesis.

(*b*) The map $\mu_{F(1),N}$ is surjective in degree 0. For each $p \geq 0$, the vector space $(TF(p))^0$ is dual to

$$\mathrm{Hom}_{\mathcal{A}}(TF(p), J(0)) \simeq \mathrm{Hom}_{\mathcal{A}}(F(p), \mathrm{H}^*(B\mathbf{F}_2)) \simeq \mathrm{H}^p(B\mathbf{F}_2).$$

In particular, it is a one-dimensional vector space over $\mathbf{F}_2$, generated by $t^p \in \mathrm{H}^*(B\mathbf{F}_2) \simeq \mathbf{F}_2[t]$. It follows that $T(F(1)) \otimes T(F(n))$ is also one-dimensional in degree 0. Moreover, in degree zero the map $\mu_{F(1),N}$ is dual to the composition

$$\mathbf{F}_2 \simeq \mathrm{Hom}_{\mathcal{A}}(T(F(1)) \otimes T(N), J(0)) \to \mathrm{Hom}_{\mathcal{A}}(T(F(1) \otimes N), J(0)) \simeq \mathrm{Hom}_{\mathcal{A}}(F(1) \otimes N, \mathrm{H}^*(B\mathbf{F}_2)).$$

We wish to show that this map is injective. For this, it suffices to observe that the image of the nontrivial element of $\mathbf{F}_2$ is a homomorphism $F(1) \otimes N \to \mathrm{H}^*(B\mathbf{F}_2)$ given by multiplying the nontrivial maps $F(1) \to \mathrm{H}^*(B\mathbf{F}_2)$ and $N \to \mathrm{H}^*(B\mathbf{F}_2)$, and that this map is nontrivial in degree $n+1$.

(*c*) The map $\mu_{F(1),N}$ is injective in degree zero. Given (*b*) and the observation that $T(F(1)) \otimes T(N)$ is one-dimensional in degree 0, it will suffice to show that the dimension of $T(F(1) \otimes N)^0$ is at most 1. We will prove the following more general assertion:

($*_p$) The dimension of $T(\Phi^p F(1) \otimes F(n))^0$ is at most 1.

For p large, we will invoke the following lemma:

Lemma 6. *Fix an integer n. Then for $p \gg 0$, the tensor product $\Phi^p F(1) \otimes F(n)$ is generated by a single element.*

Assuming Lemma 6, we deduce that for $p \gg 0$ we have a surjection $F(m) \to \Phi^p F(1) \otimes F(n)$. This induces a surjection

$$F(m) \oplus F(m-1) \oplus \ldots \oplus F(0) \simeq TF(m) \to T(\Phi^p F(1) \otimes F(n)).$$

Since the left hand side has dimension 1 in degree 0, assertion $(*_p)$ follows.

To prove $(*_p)$ in general, we use descending induction on p. We have an exact sequence

$$0 \to \Phi^{p+1} F(1) \otimes F(n) \to \Phi^p F(1) \otimes F(n) \to \Sigma^{2^p} F(n) \to 0$$

Since T is an exact functor which commutes with Σ, this reduces to an isomorphism $T(\Phi^{p+1} F(1) \otimes F(n))^0 \simeq T(\Phi^p F(1) \otimes F(n))^0$, so that $(*_{p+1})$ implies $(*_p)$ as desired.

□

We will give the proof of Lemma 6 in the next lecture.