18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The T-functor and Unstable Algebras (Lecture 20)

Our first order of business is to prove the following assertion, which was stated without proof in the previous lecture:

Lemma 1. *Fix an integer n. Then for $p \gg 0$, the tensor product $\Phi^p F(1) \otimes F(n)$ is generated by a single element.*

Proof. We may identify $F(n)$ with the subspace of $\mathbf{F}_2[t_1, \ldots, t_n]$ spanned by those polynomials which are symmetric and additive in each variable. The module $\Phi^p F(1)$ can similarly be identified with the subspace of $\mathbf{F}_2[t]$ spanned by those polynomials of the form $\{t^{2^k}\}_{k \geq p}$. We wish to show that the tensor product $\Phi^p F(1) \otimes F(n)$ is generated by the element $t^{2^p} \otimes (t_1 \ldots t_n)$. This element determines a map

$$F(n + 2^p) \rightarrow \Phi^p F(1) \otimes F(n);$$

it will therefore suffice to show that β is surjective. The right hand side has a basis consisting of expressions of the form

$$t^{2^{p+q}} \otimes \sigma(t_1^{2^{b_1}} \ldots t_n^{2^{b_n}}),$$

where σ denotes the operation of symmetrization. We now observe that this basis element is the image of

$$\sigma(t_1^{2^{b_1}} \ldots t_n^{2^{b_n}} t_{n+1}^{2^q} t_{n+2}^{2^q} \ldots t_{n+2^p}^{2^q}) \in F(n + 2^p)$$

provided that $2^p > n$. $\square$

In the last lecture, we saw that Lemma 1 implies that Lannes' T-functor T_V commutes with tensor products. It follows that M is an unstable $\mathcal{A}$-module equipped with a multiplication map $M \otimes M \rightarrow M$, then $T_V(M)$ inherits a multiplication

$$T_V M \otimes T_V M \simeq T_V(M \otimes M) \rightarrow T_V M.$$

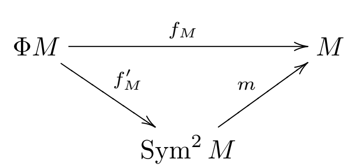
Proposition 2. *Suppose that M is an unstable $\mathcal{A}$-algebra. Then the multiplication defined above endows $T_V M$ with the structure of an unstable $\mathcal{A}$-algebra.*

Proof. Since M is commutative, associative, and unital, we deduce immediately that $T_V M$ has the same properties. The only nontrivial point is to verify that $\mathrm{Sq}^{\deg(x)}(x) = x^2$ for every homogeneous element $x \in T_V M$. Before proving this, we indulge in a slight digression.

Let M be an unstable $\mathcal{A}$-module. There is a canonical map $f'_M : \Phi M \rightarrow \mathrm{Sym}^2 M$, given by the formula

$$\Phi(x) \mapsto x^2.$$

By definition, an unstable $\mathcal{A}$-algebra is an unstable $\mathcal{A}$-module M equipped with a commutative, associative, and unital multiplication $m : M \otimes M \rightarrow M$ such that the diagram



commutes. Here $f_M : \Phi M \to M$ is the map described by the formula $x \mapsto \mathrm{Sq}^{\deg(x)}\, x$.

Applying T_V to the commutative diagram above, we get a new commutative diagram

$$\begin{array}{ccc} T_V \Phi M & \xrightarrow{T_V f_M} & T_V M \\ & {\scriptstyle T_V f'_M} \searrow \quad \nearrow {\scriptstyle T_V m} & \\ & T_V \operatorname{Sym}^2 M. & \end{array}$$

Since the functor T_V preserves colimits and tensor products, we have a canonical isomorphism $\alpha : T_V \operatorname{Sym}^2 M \simeq \operatorname{Sym}^2 T_V M$; similarly we have an identification $\beta : T_V \Phi M \simeq \Phi T_V M$. Under the isomorphism α, the map $T_V M$ corresponds to the multiplication map $\operatorname{Sym}^2 T_V M \to T_V M$ given by the ring structure on $T_V M$. To prove that $T_V M$ is an unstable $\mathcal{A}$-algebra, it will suffice to show that the maps $T_V f_M$ and $T_V f'_M$ can be identified, by means of α and β, with $f_{T_V M}$ and $f'_{T_V M}$, respectively. We will give a proof for $f'_{T_V M}$, leaving the first case as an exercise to the reader.

We wish to show that the diagram

$$\begin{array}{ccc} T_V \Phi M & \xrightarrow{T_V f'_M} & T_V \operatorname{Sym}^2 M \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \Phi T_V M & \xrightarrow{f'_{T_V M}} & \operatorname{Sym}^2 T_V M \end{array}$$

is commutative. Using the definition of T_V, we are reduced to proving that the adjoint diagram

$$\begin{array}{ccc} \Phi M & \xrightarrow{f'_M} & \operatorname{Sym}^2 M \\ \downarrow & & \downarrow \\ (\Phi T_V M) \otimes \mathrm{H}^*(BV) & \longrightarrow & (\operatorname{Sym}^2 T_V M) \otimes \mathrm{H}^*(BV). \end{array}$$

To prove this, we consider the larger diagram

$$\begin{array}{ccc} \Phi M & \xrightarrow{f'_M} & \operatorname{Sym}^2 M \\ \downarrow & & \downarrow \\ \Phi(T_V M \otimes \mathrm{H}^*(BV)) & \longrightarrow & \operatorname{Sym}^2(T_V M \otimes \mathrm{H}^*(BV)) \\ \downarrow{\scriptstyle \sim} & & \downarrow \\ \Phi(T_V M) \otimes \Phi\, \mathrm{H}^*(BV) & \longrightarrow & \operatorname{Sym}^2(T_V M) \otimes \operatorname{Sym}^2 \mathrm{H}^*(BV) \\ \downarrow & & \downarrow \\ \Phi(T_V M) \otimes \mathrm{H}^*(BV) & \longrightarrow & \operatorname{Sym}^2(T_V M) \otimes \mathrm{H}^*(BV). \end{array}$$

The top square obviously commutes. The middle square commutes because the construction of the map f'_M is compatible with the formation of tensor products in M. The lower square commutes because $\mathrm{H}^*(BV)$ is an unstable $\mathcal{A}$-algebra. It follows that the outer square commutes as well, as desired. $\square$

Let M be an unstable $\mathcal{A}$-algebra, so that $T_V M$ inherits the structure of an unstable $\mathcal{A}$-algebra. We now characterize $T_V M$ by a universal property.

Proposition 3. *Let $\mathcal{K}$ denote the category of unstable $\mathcal{A}$-algebras. For every pair of objects $M, N \in \mathcal{K}$, the image of the inclusion*

$$\mathrm{Hom}_{\mathcal{K}}(T_V M, N) \subseteq \mathrm{Hom}_{\mathcal{A}}(T_V M, N) \simeq \mathrm{Hom}_{\mathcal{A}}(M, N \otimes \mathrm{H}^*(BV))$$

consists of those maps $M \to N \otimes \mathrm{H}^*(BV)$ which are compatible with the ring structure.

Proof. We will show that a map $u : T_V M \to N$ is compatible with multiplication if and only if the adjoint map $v : M \to N \otimes \mathrm{H}^*(BV)$ is compatible with multiplication; an analogous (but easier) argument shows that u is unital if and only if v is unital.

By definition, u is compatible with multiplication if and only if the diagram

$$\begin{array}{ccc} (T_V M) \otimes (T_V M) & \xrightarrow{u \otimes u} & N \otimes N \\ \uparrow & & \\ T_V(M \otimes M) & & \Big\downarrow \\ \downarrow & & \\ T_V M & \xrightarrow{u} & N \end{array}$$

is commutative. This is equivalent to the commutativity of the adjoint diagram

$$\begin{array}{ccc} T_V M \otimes T_V M \otimes \mathrm{H}^*(BV) & \xrightarrow{w_1} & N \otimes N \otimes \mathrm{H}^*(BV) \\ \uparrow{\scriptstyle w_0} & & \\ M \otimes M & & \Big\downarrow{\scriptstyle w_2} \\ \downarrow & & \\ M & \xrightarrow{v} & N \otimes \mathrm{H}^*(BV). \end{array}$$

To prove that this is equivalent to the assumption that v is compatible with multiplication, it will suffice to show that the composition $w_2 \circ w_1 \circ w_0$ coincides with the composition

$$M \otimes M \stackrel{v \otimes v}{\to} (N \otimes \mathrm{H}^*(BV)) \otimes (N \otimes \mathrm{H}^*(BV)) \to N \otimes \mathrm{H}^*(BV).$$

This follows from the commutativity of the diagram

$$\begin{array}{ccc} M \otimes M & \xrightarrow{v \otimes v} & \searrow \\ \downarrow & & \\ (T_V M \otimes \mathrm{H}^*(BV)) \otimes (T_V M \otimes \mathrm{H}^*(BV)) & \xrightarrow{u \otimes u} & (N \otimes \mathrm{H}^*(BV)) \otimes (N \otimes \mathrm{H}^*(BV)) \\ \downarrow & & \downarrow \\ T_V M \otimes T_V M \otimes \mathrm{H}^*(BV) & \longrightarrow & N \otimes N \otimes \mathrm{H}^*(BV) \\ & & \downarrow \\ & & N \otimes \mathrm{H}^*(BV). \end{array}$$

□

Corollary 4. *Regarded as a functor from $\mathcal{K}$ to itself, Lannes' T-functor is left adjoint to the functor $N \mapsto N \otimes \mathrm{H}^*(BV)$.*

Corollary 5. *Let $F_{\mathrm{Alg}}(n)$ denote the free unstable $\mathcal{A}$-algebra on one generator in degree n. Then we have a canonical isomorphism of unstable $\mathcal{A}$-algebras*

$$TF_{\mathrm{Alg}}(n) \simeq F_{\mathrm{Alg}}(n) \otimes \ldots \otimes F_{\mathrm{Alg}}(0).$$

Proof. Let M be an arbitrary unstable $\mathcal{A}$-algebra. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}}(TF_{\mathrm{Alg}}(n), M) &\simeq \mathrm{Hom}_{\mathcal{K}}(F_{\mathrm{Alg}}(n), M \otimes \mathbf{F}_2[t]) \\ &\simeq (M \otimes \mathbf{F}_2[t])^n \\ &\simeq M^n \times M^{n-1} \times \ldots \times M^0 \\ &\simeq \mathrm{Hom}_{\mathcal{K}}(F_{\mathrm{Alg}}(n), M) \times \ldots \times \mathrm{Hom}_{\mathcal{K}}(F_{\mathrm{Alg}}(0), M) \\ &\simeq \mathrm{Hom}_{\mathcal{K}}(F_{\mathrm{Alg}}(n) \otimes \ldots \otimes F_{\mathrm{Alg}}(0), M). \end{aligned}$$

□

Recall that $F_{\mathrm{Alg}}(n)$ can be identified with the cohomology of the Eilenberg-MacLane space $K(\mathbf{F}_2, n)$. Similarly, the Kunneth theorem allows us to identify the tensor product $F_{\mathrm{Alg}}(n) \otimes \ldots \otimes F_{\mathrm{Alg}}(0)$ with the cohomology of the product

$$K(\mathbf{F}_2, n) \times K(\mathbf{F}_2, n-1) \times \ldots \times K(\mathbf{F}_2, 0) \simeq K(\mathbf{F}_2, n)^{B\mathbf{F}_2}.$$

The isomorphism of Corollary 5 is induced by the canonical map

$$\eta_X : T_V \, \mathrm{H}^*(X) \to \mathrm{H}^*(X^{BV})$$

in the special case where $X = K(\mathbf{F}_2, n)$ and $V = \mathbf{F}_2$. We may therefore restate Corollary 5 in the following more conceptual form: if X is an Eilenberg-MacLane space $K(\mathbf{F}_2, n)$ and $V = \mathbf{F}_2$, then the map η_X is an isomorphism. Our next goal in this course is to prove this statement for a much larger class of spaces.