18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

A Pushout Square (Lecture 22)

In the last lecture we saw that the cohomology $\mathrm{H}^* \mathcal{F}(n)$ of the free E_∞-algebra on one generator was itself freely generated by one element, as an unstable algebra over the big Steenrod algebra $\mathcal{A}^{\mathrm{Big}}$. The Cartan-Serre theorem implies that the cohomology ring $\mathrm{H}^* K(\mathbf{F}_2, n)$ is the free unstable $\mathcal{A}$-module on one generator, in the same degree. This suggests a close relationship between $\mathrm{H}^* \mathcal{F}(n)$ and $\mathrm{H}^* K(\mathbf{F}_2, n)$. In fact, we can say more: there is a close relationship between the E_∞-algebras $\mathcal{F}(n)$ and $C^* K(\mathbf{F}_2, n)$ for each $n \geq 0$.

To make this precise, we begin by observing that the canonical element $\nu \in \mathrm{H}^n K(\mathbf{F}_2, n)$ gives rise to a map of E_∞-algebras

$$f : \mathcal{F}(n) \to C^* K(\mathbf{F}_2, n).$$

Let μ denote the canonical generator of $\mathrm{H}^* \mathcal{F}(n)$, so that f carries μ to ν.

The map f is certainly not a homotopy equivalence. The target $\mathrm{H}^* K(\mathbf{F}_2, n)$ is a module over the usual Steenrod algebra $\mathcal{A}$, so that Sq^0 acts by the identity on $\mathrm{H}^* K(\mathbf{F}_2, n)$. However, Sq^0 does not act by the identity on the cohomology of the left hand side. We therefore have

$$f(\mu - \mathrm{Sq}^0 \mu) = f(\mu) - \mathrm{Sq}^0 f(\mu) = \nu - \mathrm{Sq}^0 \nu = 0,$$

so that f fails to be injective on cohomology.

However, this turns out to be the *only* obstruction to f being a homotopy equivalence. To make this precise, we observe that there is map $g : \mathcal{F}(n) \to \mathcal{F}(n)$, which is determined up to homotopy by the requirement that $g(\mu) = \mu - \mathrm{Sq}^0 \mu \in \mathrm{H}^n \mathcal{F}(n)$. The above calculation shows that $f \circ g$ carries μ to zero in $\mathrm{H}^n K(\mathbf{F}_2, n)$. We therefore obtain a (homotopy) commutative diagram of E_∞-algebras

$$\begin{array}{ccc} \mathcal{F}(n) & \xrightarrow{g} & \mathcal{F}(n) \\ \downarrow & & \downarrow f \\ \mathbf{F}_2 & \longrightarrow & C^* K(\mathbf{F}_2, n). \end{array}$$

Our goal in this lecture is to prove:

Theorem 1. *The above diagram is a homotopy pushout square in the category of E_∞-algebras over $\mathbf{F}_2$.*

In other words, the cochain complex $C^* K(\mathbf{F}_2, n)$ has a very simple presentation as an E_∞-algebra over $\mathbf{F}_2$. It is "generated" by the tautological class $\nu \in \mathrm{H}^n K(\mathbf{F}_2, n)$, and subject only to the "relation" that ν is fixed by Sq^0.

To prove Theorem 1, we need to understand homotopy pushouts in the world of E_∞-algebras. We first recall the situation for ordinary commutative rings. Given a pair of commutative ring homomorphisms

$$A \leftarrow R \rightarrow B,$$

the pushout $A \coprod_R B$ in the category of commutative rings is given by the relative tensor product $A \otimes_R B$. In the case of E_∞-algebras, the situation is more or less identical. More precisely:

- Given an E_∞-algebra R, there is a good theory of R*-modules* (or R*-module spectra*).
- Given any map $R \to A$ of E_∞-algebras, we can regard A as an R-module.
- Given an E_∞-ring R, the collection of R-module spectra is endowed with a tensor product operation $(M, N) \mapsto M \otimes_R N$. (More traditionally, this is denoted by $M \wedge_R N$ and called the *smash product over* R).
- Given a pair of E_∞-algebra maps
$$A \leftarrow R \rightarrow B,$$
the homotopy pushout of A and B over R in the setting of E_∞-rings is again an R-algebra, and the underlying R-module is given by the tensor product $A \otimes_R B$.

Given these facts, we can restate Theorem 1. We have a canonical map

$$\mathcal{F}(n) \otimes_{\mathcal{F}(n)} \mathbf{F}_2 \to C^* K(\mathbf{F}_2, n),$$

and we wish to show that this map is a homotopy equivalence. In other words, we wish to show that it induces an isomorphism after passing to cohomology. The cohomology of the right side is given by the Cartan-Serre theorem: $\mathrm{H}^* K(\mathbf{F}_2, n)$ can be identified with the polynomial ring on generators $\{\mathrm{Sq}^I \nu\}$, where I ranges over admissible positive sequences of excess $< n$. It therefore remains to compute the cohomology of the left hand side.

The calculation will be based on the following lemma:

Lemma 2. *Let R be an E_∞-algebra over $\mathbf{F}_2$, and let M and N be R-modules. Then $\mathrm{H}^* M$ and $\mathrm{H}^* N$ are modules over the cohomology ring $\mathrm{H}^* R$. Suppose that $\mathrm{H}^* M$ is* free *as a graded $\mathrm{H}^* R$-module. Then the canonical map*

$$\mathrm{H}^* M \otimes_{\mathrm{H}^* R} \mathrm{H}^* N \to \mathrm{H}^*(M \otimes_R N)$$

is an isomorphism.

Proof. Choose elements $\{x_i \in \mathrm{H}^{n_i} M\}$ which freely generate $\mathrm{H}^* M$ as an $\mathrm{H}^* R$-module. Each x_i determines a map of R-modules $R[-n_i] \to M$. Adding these together, we obtain a map $\oplus R[-n_i] \to M$. By assumption this map induces an isomorphism on cohomology, and is therefore a homotopy equivalence. Thus, M is a direct sum of *free* R-modules (in various degrees).

Let us say that an R-module M is *good* if the canonical map

$$\mathrm{H}^* M \otimes_{\mathrm{H}^* R} \mathrm{H}^* N \to \mathrm{H}^*(M \otimes_R N)$$

is an isomorphism. Both the left hand side and the right hand side above are functors of M, which commute with shifting and with the formation of direct sums. Therefore, to show that $\oplus R[-n_i]$ is good, it will suffice to show that R is good. But this is clear, since

$$\mathrm{H}^* R \otimes_{\mathrm{H}^* R} \mathrm{H}^* N \simeq \mathrm{H}^* N \simeq \mathrm{H}^*(R \otimes_R N).$$

□

To prove Theorem 1, we will show that Lemma 2 applies: namely, that $\mathrm{H}^* \mathcal{F}(n)$ is *free* when regarded s an $\mathrm{H}^* \mathcal{F}(n)$-module via the map g. It then follows that we have an isomorphism

$$\mathrm{H}^*(\mathcal{F}(n) \otimes_{\mathcal{F}(n)} \mathbf{F}_2) \simeq \mathrm{H}^* \mathcal{F}(n) \otimes_{\mathrm{H}^* \mathcal{F}(n)} \mathbf{F}_2 = \mathrm{H}^* \mathcal{F}(n)/I,$$

where I is the ideal of $\mathrm{H}^* \mathcal{F}(n)$ generated by the elements $g(x)$, where $x \in \mathrm{H}^* \mathcal{F}(n)$ has positive degree.

In the last lecture, we proved that $\mathrm{H}^* \mathcal{F}(n)$ is isomorphic to the free unstable $\mathcal{A}^{\mathrm{Big}}$-module $F^{\mathrm{Big}}_{\mathrm{Alg}}(n)$. It is therefore isomorphic to a polynomial ring on generators $\{\mathrm{Sq}^I \mu\}$, where I ranges over admissible sequences of excess $< n$. For every such sequence I, we let $X_I = g(\mathrm{Sq}^I \mu) = \mathrm{Sq}^I \mu - \mathrm{Sq}^I \mathrm{Sq}^0 \mu \in \mathrm{H}^* \mathcal{F}(n)$. To complete the proof of Theorem 1, it will suffice to verify the following:

Proposition 3. *The cohomology ring* $\mathrm{H}^* \mathcal{F}(n)$ *is a polynomial ring on generators* $\{X_I\}_{I \, admissible \; of \; excess \; <n}$ *and* $\{\mathrm{Sq}^I \mu\}_{I \, admissible \; and \; positive \; of \; excess \; <n}$.

Proof. Let $\mathcal{J}$ denote the collection of all admissible sequences of integers of excess $< n$. We have a decomposition $\mathcal{J} = \mathcal{J}' \coprod \mathcal{J}''$, where $\mathcal{J}'$ consists of those sequences $(i_1, \ldots, i_k)$ such that $k > 0$ and $i_k < 0$. The complement $\mathcal{J}''$ has a further decomposition

$$\mathcal{J}'' = \mathcal{J}''(0) \coprod \mathcal{J}''(1) \coprod \cdots$$

where $\mathcal{J}''(m)$ consists of those sequence $(i_1, \ldots, i_k)$ which end with precisely k zeroes. For each $I \in \mathcal{J}''(k)$, let $I^+ \in \mathcal{J}''(k+1)$ be the result of appending a zero to the sequence I. We have a decomposition

$$\mathrm{H}^* \mathcal{F}(n) \simeq \mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}'} \otimes \mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}''}.$$

To complete the proof, it will suffice to show:

(1) The polynomial ring $\mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}'}$ is also polynomial on the generators $\{X_I\}_{I \in \mathcal{J}'}$.

(2) The polynomial ring $\mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}''}$ is also polynomial on the generators $\{X_I\}_{I \in \mathcal{J}''}$ and $\{\mathrm{Sq}^I \mu\}_{I \in \mathcal{J}''(0)}$.

Assertion (2) follows immediately from the observation that $X_I = \mathrm{Sq}^I \mu - \mathrm{Sq}^{I^+} \mu$ for $I \in \mathcal{J}''$. We can divide the proof of (1) further into three steps:

($1a$) The map $\theta : \mathbf{F}_2[X_I]_{I \in \mathcal{J}'} \to \mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}'}$ is well-defined. In other words, if $I \in \mathcal{J}'$, then X_I belongs to $\mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}'}$.

($1b$) The map θ is injective.

($1c$) The map θ is surjective.

Assertion ($1a$) is an immediate consequence of the following:

Lemma 4. *Let* $I = (i_m, \ldots, i_1)$ *be a sequence of integers with* $i_1 < 0$. *Then in* $\mathcal{A}^{Big}$ *we have an equality*

$$\mathrm{Sq}^I \mathrm{Sq}^0 = \sum_\alpha \mathrm{Sq}^{J_\alpha}$$

where each J_α *is an admissible sequence of the form* $(j_m, \ldots, j_0)$*, where* $j_0 < 0$.

Proof. We first apply the Adem relations to write

$$\mathrm{Sq}^{i_1} \mathrm{Sq}^0 = \sum_k (2k - i_1, -k-1) \mathrm{Sq}^k \mathrm{Sq}^{i_1 - k}.$$

The coefficient $(2k - i_1, -k-1)$ vanishes unless

$$\frac{i_1}{2} \leq k < 0.$$

We may therefore restrict our attention to those integers k for which $i_1 - k \leq \frac{i_1}{2} < 0$, so the sequence $I'(k) = (i_m, \ldots, i_2, k, i_1 - k)$ ends with a negative integer.

Each $I'(k)$ can be rewritten as a sum of admissible monomials using the Adem relations. Let us analyze this process. Given a sequence

$$J = (j_m, \ldots, a, b, \ldots, j_0)$$

with $a < 2b$, we have

$$\mathrm{Sq}^J = \sum_k (2k - a, b-k-1) \mathrm{Sq}^{J_k},$$

where J_k is obtained from J by replacing a by $b+k$ and b by $a-k$. The coefficient $(2k-a, b-k-1)$ vanishes unless $\frac{a}{2} \leq k < b$; in particular, we always have $a - k \leq \frac{a}{2} < b$. Thus, if the final entry in J is negative, the final entry in J_k will be negative. □

We now prove (1b). Recall that the cohomology ring $\mathrm{H}^* \mathcal{F}(n) \simeq \mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}}$ has a natural grading by rank, where $\mathrm{Sq}^I \mu$ has rank 2^k for every sequence $I = (i_1, \ldots, i_k)$. This grading restricts to a grading on $\mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}'}$. We have an analogous grading on $\mathbf{F}_2[X_I]_{I \in \mathcal{J}'}$, where we declare $\mathrm{rk}(X_I) = 2^k$ if $I = (i_1, \ldots, i_k)$.

The map $\theta : \mathbf{F}_2[X_I]_{I \in \mathcal{J}'} \to \mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}'}$ is not compatible with the gradings by rank. Instead we have

$$\theta(X_I) = \mathrm{Sq}^I \mu - \mathrm{Sq}^I \mathrm{Sq}^0 \mu = \mathrm{Sq}^I \mu + \text{ higher rank.}$$

We have an evident isomorphism $\theta' : \mathbf{F}_2[X_I]_{I \in \mathcal{J}'} \to \mathbf{F}_2[\mathrm{Sq}^I \mu]_{I \in \mathcal{J}'}$, given by $X_I \mapsto \mathrm{Sq}^I \mu$. Let $x \in \mathbf{F}_2[X_I]_{I \in \mathcal{J}'}$ be a nonzero element, and write x as a sum $x = x_{k_0} + x_{k_1} + \ldots + x_{k_m}$ of homogeneous elements of ranks $k_0 < k_1 < \ldots < k_m$. Then we have

$$\theta(x) = \theta'(x) + \text{ terms of rank ¿ k.}$$

In particular, $\theta(x) = 0$ implies $\theta'(x_{k_0}) = 0$. Since θ' is an isomorphism, we get $x_{k_0} = 0$, a contradiction. This completes the proof that θ is injective.

We now prove that θ is surjective. This is an immediate consequence of the following statement:

Lemma 5. *Let $I = (i_k, \ldots, i_1)$ be a sequence of integers with $i_1 < 0$ (not necessarily admissible). Then $\mathrm{Sq}^I \mu$ lies in the image of θ.*

Proof. We use descending induction on i_1. Observe that

$$\mathrm{Sq}^I \mu = (\mathrm{Sq}^I \mu - \mathrm{Sq}^I \mathrm{Sq}^0 \mu) + (\mathrm{Sq}^I \mathrm{Sq}^0 \mu) = \theta(X_I) + \mathrm{Sq}^I \mathrm{Sq}^0 \mu.$$

It will therefore suffice to show that $\mathrm{Sq}^I \mathrm{Sq}^0 \mu$ belongs to the image of θ. Using the Adem relations, we can write

$$\mathrm{Sq}^I \mathrm{Sq}^0 = \sum_k (2k - i_1, -k-1) \mathrm{Sq}^{I_k}$$

with $I_k = (i_k, \ldots, i_2, k, i_1 - k)$. The coefficient $(2k - i_1, -k-1)$ vanishes unless $\frac{i_1}{2} \leq k < 0$. This inequality forces

$$i_1 < i_1 - k \leq \frac{i_1}{2} < 0.$$

Therefore Sq^{I_k} belongs to the image of θ by the inductive hypothesis. □

□

Corollary 6. *For each $n \geq 0$, the homotopy pullback square*

$$\begin{array}{ccc} K(\mathbf{F}_2, n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & K(\mathbf{F}_2, n+1) \end{array}$$

of topological spaces determines a homotopy pushout square

$$\begin{array}{ccc} C^* K(\mathbf{F}_2, n) & \longleftarrow & \mathbf{F}_2 \\ \uparrow & & \uparrow \\ \mathbf{F}_2 & \longleftarrow & C^* K(\mathbf{F}_2, n+1) \end{array}$$

of E_∞-algebras.

Proof. Theorem 1 implies that $C^*K(\mathbf{F}_2, n+1)$ is freely generated by a single class ν in degree $(n+1)$, subject to the single relation killing $\nu - \mathrm{Sq}^0\,\nu$. We can regard the homotopy pushout

$$\mathbf{F}_2 \otimes_{C^*K(\mathbf{F}_2,n+1)} \mathbf{F}_2$$

as the *suspension* of $C^*K(\mathbf{F}_2, n+1)$ in the world of (augmented) E_∞-algebras. Consequently, it has an analogous presentation as the free E_∞-algebra generated by a class $\Sigma(\nu)$ in degree n, subject to a single relation killing $\Sigma(\nu - \mathrm{Sq}^0\,\nu)$. Since the Steenrod operation Sq^0 is stable, we can identify $\Sigma(\nu - \mathrm{Sq}^0\,\nu)$ with $\Sigma(\nu) - \mathrm{Sq}^0\,\Sigma(\nu)$. Applying Theorem 1 again, we can identify this suspension with $C^*K(\mathbf{F}_2, n)$. It is easy to see that this identification is given by the map

$$\mathbf{F}_2 \otimes_{C^*K(\mathbf{F}_2,n+1)} \mathbf{F}_2 \to C^*K(\mathbf{F}_2, n)$$

described in the statement of Corollary 6. □