18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

T and the Cohomology of Spaces (Lecture 25)

In the last lecture, we showed that if G denotes the forgetful functor from the category of E_∞-algebras over $\mathbf{F}_2$ to spectra, then $R = \mathrm{Map}(G, G)$ is an A_∞-ring spectrum whose homotopy groups $\pi_* R$ form a graded ring, isomorphic to a suitable completion of the big Steenrod algebra $\mathcal{A}^{\mathrm{Big}}$.

Remark 1. If A is an E_∞-algebra over $\mathbf{F}_2$, then A is in particular an $\mathbf{F}_2$-module, so that $\mathbf{F}_2$ acts on the underlying spectrum of A. This construction is functorial in A, and so gives rise to a map of A_∞-algebras from $\mathbf{F}_2$ into R. This map is *not* central. That is, R is an A_∞-ring spectrum, but it cannot be regarded as an A_∞-algebra over the ring $\mathbf{F}_2$.

This result has an analogue for the ordinary Steenrod algebra. More precisely, let $R' = \mathrm{Map}(\mathbf{F}_2, \mathbf{F}_2)$ be the A_∞-algebra of endomorphisms of the Eilenberg-MacLane spectrum $H\mathbf{F}_2$. Then R' can be identified with the homotopy inverse limit of reduced cochain complexes

$$\operatorname{proj}\lim \overline{C}^*(K(\mathbf{F}_2, n); \mathbf{F}_2)[n],$$

so we get short exact sequences

$$0 \to \lim^1\{\mathrm{H}^{n+k+1} K(\mathbf{F}_2, n)\} \to \pi_{-k} R' \to \lim\{\mathrm{H}^{n+k} K(\mathbf{F}_2, n)\} \to 0.$$

Using the same argument as in the previous lecture, we deduce that the $\lim^1$-term vanishes, and the right hand side can be identified with the inverse limit of vector spaces having basis $\{\mathrm{Sq}^I \mu_n\}$, where I ranges over positive admissible monomials of degree k and excess $\leq n$. This sequence of vector spaces stabilizes, since every positive admissible sequence $I = (i_1, \ldots, i_m)$ has excess $i_1 - i_2 - \ldots - i_m \leq i_1 + i_2 + \ldots + i_m = \deg(I)$. Passing to the inverse limit, we get an isomorphism of graded rings

$$\pi_* R' \simeq \mathcal{A}\,.$$

By construction, R acts on the underlying spectrum of every E_∞-algebra over $\mathbf{F}_2$. In particular, R acts on $\mathbf{F}_2$ itself, via a map $R \to R'$ which induces, on the level of homotopy groups, the canonical surjection $\mathcal{A}^{\mathrm{Big}} \to \mathcal{A}$.

We now turn to the real goal of this lecture. Let X be a topological space, and V a finite dimensional vector space over $\mathbf{F}_2$. We have a canonical evaluation map

$$X^{BV} \times BV \to X$$

which induces on cohomology a map

$$\mathrm{H}^* X \to \mathrm{H}^*(X^{BV} \times BV) \simeq \mathrm{H}^* X^{BV} \otimes \mathrm{H}^* BV.$$

This is adjoint to a map

$$\theta_X : T_V \, \mathrm{H}^* X \to \mathrm{H}^* X^{BV}$$

of unstable $\mathcal{A}$-algebras. We will prove:

Theorem 2. *Suppose that X is a 2-finite space. Then the map θ_X is an isomorphism.*

Remark 3. If X is 2-finite, then any mapping space X^{BV} is again 2-finite. To see this, we first use induction on V to reduce to the case where $V \simeq \mathbf{F}_2$. Choose a filtration $X \simeq X_m \to \ldots \to X_0 \simeq *$, where each map is a fibration whose fiber is an Eilenberg-MacLane space $K(\mathbf{F}_2, n)$. Then we have an induced filtration

$$X^{B\mathbf{F}_2} \simeq X_m^{B\mathbf{F}_2} \to \ldots \to X_0^{B\mathbf{F}_2} \simeq *,$$

and each map is a fibration whose fiber is a generalized Eilenberg-MacLane space $K(\mathbf{F}_2, n) \times K(\mathbf{F}_2, n-1) \times \ldots \times K(\mathbf{F}_2, 0)$ (and in particular 2-finite).

We have already proven Theorem 2 in the case where $V = \mathbf{F}_2$ and X is an Eilenberg-MacLane space $K(\mathbf{F}_2, n)$. It follows, by induction on the dimension of V, that Theorem 2 holds in general when $X = K(\mathbf{F}_2, n)$. (It is also possible to prove this by repeating the original argument.)

If X is a disjoint union of path components X_α (necessarily finite in number), then θ_X can be identified with the product of the maps θ_{X_α}. Therefore, to prove Theorem 2 it suffices to treat the case where X is path connected. In this case, we have seen that X admits a finite filtration

$$X \simeq X_m \to X_{m-1} \to \ldots \to X_0 \simeq *$$

where each X_{i+1} is a principal fibration over X_i with fiber $K(\mathbf{F}_2, n_i)$. We will prove that each θ_{X_i} is an isomorphism, using induction on i: the case $i = 0$ is obvious. To handle, the inductive step, we study the homotopy pullback square

$$\begin{array}{ccc} X_{i+1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & K(\mathbf{F}_2, n_i + 1). \end{array}$$

It will suffice to prove the following:

Proposition 4. *Suppose given a homotopy pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

of 2-finite spaces. If θ_X, θ_Y, and $\theta_{Y'}$ are isomorphisms, then so is $\theta_{X'}$.

We begin with a few general remarks. Let A be an E_∞-algebra over $\mathbf{F}_2$, and let M and N be a pair of A-modules. The relative tensor product $M \otimes_A N$ is defined to be the geometric realization of a simplicial spectrum $B^A_\bullet(M, N)$, with

$$B^A_n(M, N) = M \otimes A \otimes \ldots \otimes A \otimes N$$

(here the factor A appears n-times, and all tensor products are taken over $\mathbf{F}_2$).

For any simplicial spectrum $X_\bullet$, the homotopy groups of the geometric realization $|X_\bullet|$ can be computed by means of a spectrum sequence with E_1 term given by

$$E_1^{p,q} = \pi_p X_q.$$

If R is an A_∞-algebra, and $X_\bullet$ is a simplicial R-module spectrum, then this spectral sequence is a spectral sequence of $\pi_* R$-modules: that is, for each $1 \leq r \leq \infty$ we have maps

$$E_r^{p,q} \otimes \pi_{p'} R \to E_r^{p+p',q}$$

which exhibit each $E_r^{*,q}$ as a module over $\pi_* R$, and the differentials are compatible with this module structure.

In particular, suppose that A is an E_∞-algebra over $\mathbf{F}_2$, and that M and N are E_∞-algebras over A. Then the simplicial object $B_n^A(M, N)$ is a simplicial E_∞-algebra over $\mathbf{F}_2$, and in particular a simplicial R-module, where R is the ring spectrum studied in the previous lecture. It follows that the homotopy groups $\pi_*(M \otimes_A N)$ can be computed by a spectral sequence $\{E_r^{p,q}, d_r\}$ satisfying the following:

(a) Each $E_r^{*,q}$ is a module over the big Steenrod algebra $\mathcal{A}^{\mathrm{Big}}$.

(b) Each differential d_r is compatible with the action of $\mathcal{A}^{\mathrm{Big}}$.

(c) Each $E_1^{*,q}$ is isomorphic (as an $\mathcal{A}^{\mathrm{Big}}$-module) to the tensor product

$$\pi_* M \otimes \pi_* A \otimes \ldots \otimes \pi_* A \otimes \pi_* N,$$

where the factor $\pi_* A$ occurs q times.

We now return to the situation of Proposition 4. The convergence result of the previous lecture guarantees that the natural map

$$C^* Y' \otimes_{C^* Y} C^* X \to C^* X'$$

is an equivalence. It follows that $\mathrm{H}^* X'$ can be computed by a spectral sequence $\{E_r^{p,q}, d_r\}$ satisfying conditions (a) and (b), with

$$E_1^{-*,q} = \mathrm{H}^* Y' \otimes \mathrm{H}^* Y \otimes \ldots \otimes \mathrm{H}^* Y \otimes \mathrm{H}^* X.$$

It follows that each of the $\mathcal{A}^{\mathrm{Big}}$-modules $E_1^{-*,q}$ is actually an unstable $\mathcal{A}$-module. Since this condition is stable under passage to subquotients, we obtain the following stronger version of condition (a):

(a') Each $E_r^{*,q}$ is an unstable $\mathcal{A}$-module.

We have another homotopy pullback diagram

$$\begin{array}{ccc} X'^{BV} & \longrightarrow & X^{BV} \\ \downarrow & & \downarrow \\ Y'^{BV} & \longrightarrow & Y^{BV}, \end{array}$$

which consists of 2-finite spaces in virtue of Remark 3. Applying the same reasoning, we get another spectral sequence $\{E'^{p,q}_r, d'_r\}$ satisfying (a') and (b), with

$$E'^{-*,q}_1 \simeq \mathrm{H}^* Y'^{BV} \otimes \mathrm{H}^* Y^{BV} \otimes \ldots \otimes \mathrm{H}^* Y^{BV} \otimes \mathrm{H}^* X^{BV}.$$

The evaluation maps $Z^{BV} \times BV \to Z$ give rise to a collection of maps

$$E_r^{*,q} \to E'^{*,q}_r \otimes \mathrm{H}^* BV.$$

Passing to adjoints and using the exactness of T_V, we get a map of spectral sequences

$$T_V E_r^{*,q} \to E'^{*,q}_r.$$

Since T_V is compatible with tensor products, our hypothesis on Y', Y and X guarantees that these maps are isomorphisms when $r = 1$. It then follows by induction on r that these maps are isomorphisms for all $r < \infty$. For $r > q$, we have a sequence of surjections

$$E_r^{*,q} \to E_{r+1}^{*,q} \to \ldots$$

$$E'^{*,q}_{r} \to E'^{*,q}_{r+1} \to \dots$$

Since T_V commutes with colimits (being a left adjoint, we conclude by passing to the limit that the map $T_V E^{*,q}_\infty \to E'^{*,q}_\infty$ is an isomorphism.

We now consider the canonical map

$$T_V \operatorname{H}^* X' \to \operatorname{H}^* X'^{BV}.$$

The preceding spectral sequences give increasing filtrations

$$0 \subseteq F_0 \operatorname{H}^* X' \subseteq F_1 \operatorname{H}^* X' \subseteq \dots \subseteq \operatorname{H}^* X'$$

$$0 \subseteq F_0 \operatorname{H}^* X'^{BV} \subseteq F_1 \operatorname{H}^* X'^{BV} \subseteq \dots \subseteq \operatorname{H}^* X'^{BV}$$

by $\mathcal{A}$-submodules. Using the exactness of T_V, we get a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_V F_{i-1} \operatorname{H}^* X' & \longrightarrow & T_V F_i \operatorname{H}^* X' & \longrightarrow & T_V E^{*,i}_\infty & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_{i-1} \operatorname{H}^* X'^{BV} & \longrightarrow & F_i \operatorname{H}^* X'^{BV} & \longrightarrow & E'^{*,i}_\infty & \longrightarrow & 0. \end{array}$$

Using induction on i and the snake Lemma, we deduce that each of the maps

$$T_V F_i \operatorname{H}^* X' \to F_i \operatorname{H}^* X'^{BV}$$

is an isomorphism. Passing to the limit over i (and using the fact that T_V commutes with direct limits), we deduce that $\theta_{X'} : T_V \operatorname{H}^* X' \to \operatorname{H}^* X'^{BV}$ is an isomorphism, as desired.