18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Profinite Spaces (Lecture 26)

Let p be a prime number. In this lecture we will introduce the category of p-profinite spaces. We begin by reviewing an example from classical algebra.

Let $\mathcal{C}$ be the category of abelian groups, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory consisting of *finitely generated* abelian groups. Every abelian group A is the union of its finitely generated subgroups. Consequently, every object of $\mathcal{C}$ can be obtained as a (filtered) direct limit of objects in $\mathcal{C}_0$. Moreover, the morphisms in $\mathcal{C}$ are determined by the morphisms in $\mathcal{C}_0$. If A is a finitely generated abelian group and $\{B_\beta\}$ is any filtered system of abelian groups, then we have a bijection

$$\varinjlim \operatorname{Hom}(A, B_\beta) \operatorname{Hom}(A, \varinjlim B_\beta).$$

More generally, if A is given as a filtered colimit of abelian groups, then we get a bijection

$$\operatorname{Hom}(\varinjlim A_\alpha, \varinjlim B_\beta) \simeq \varprojlim_\alpha \operatorname{Hom}(A_\alpha, \varinjlim B_\beta) \simeq \varprojlim_\alpha \varinjlim_\beta \operatorname{Hom}(A_\alpha, B_\beta).$$

We can summarize the situation by saying that $\mathcal{C}$ is equivalent to the category of Ind-*objects* of $\mathcal{C}_0$:

Definition 1. Let $\mathcal{C}_0$ be a category. The category $\operatorname{Ind}(\mathcal{C}_0)$ of Ind-*objects* of $\mathcal{C}_0$ is defined as follows:

(1) The objects of $\operatorname{Ind}(\mathcal{C}_0)$ are *formal* direct limits "$\varinjlim C_\alpha$", where $\{C_\alpha\}$ is a filtered diagram in $\mathcal{C}_0$.

(2) Morphisms in $\operatorname{Ind}(\mathcal{C}_0)$ are given by the formula

$$\operatorname{Hom}(\text{“}\varinjlim C_\alpha\text{”}, \text{“}\varinjlim D_\beta\text{”}) = \varprojlim_\alpha \varinjlim_\beta \operatorname{Hom}(C_\alpha, D_\beta).$$

Remark 2. There is a fully faithful embedding from $\mathcal{C}_0$ into $\operatorname{Ind}(\mathcal{C}_0)$, which carries an object $C \in \mathcal{C}_0$ to the constant diagram consisting of the single object C. We will generally abuse notation and identify $\mathcal{C}_0$ with its image under this embedding.

The category $\operatorname{Ind}(\mathcal{C}_0)$ admits filtered colimits. Moreover, an object "$\varinjlim C_\alpha$" in $\operatorname{Ind}(\mathcal{C}_0)$ actually does coincide with the colimit of the diagram $\{C_\alpha\}$ in $\operatorname{Ind}(\mathcal{C}_0)$.

Remark 3. The category $\operatorname{Ind}(\mathcal{C}_0)$ can be characterized by the following universal property: for any category $\mathcal{D}$ which admits filtered colimits, the restriction functor

$$\operatorname{Fun}_0(\operatorname{Ind}(\mathcal{C}_0), \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}_0, \mathcal{D})$$

is an equivalence of categories, where the left side is the category of functors from $\operatorname{Ind}(\mathcal{C}_0)$ to $\mathcal{D}$ which preserve filtered colimits.

Example 4. Let $\mathcal{C}$ be the category of groups (or rings, or any other type of algebraic structure). Then $\mathcal{C}$ is equivalent to $\operatorname{Ind}(\mathcal{C}_0)$, where $\mathcal{C}_0 \subseteq \mathcal{C}$ is the full subcategory spanned by the finitely presented groups (or rings, etcetera).

There is a dual construction, which replaces a category $\mathcal{C}_0$ by the category $\mathrm{Pro}(\mathcal{C}_0)$ of *pro-objects* in $\mathcal{C}_0$: that is, formal inverse limits "$\varprojlim C_\alpha$" of filtered diagrams in $\mathcal{C}_0$.

Example 5. Let $\mathcal{C}_0$ be the category of *finite* groups. Then $\mathrm{Pro}(\mathcal{C}_0)$ is equivalent to the category of *profinite groups*: that is, topological groups which are compact, Hausdorff, and totally disconnected.

The construction $\mathcal{C}_0 \mapsto \mathrm{Pro}(\mathcal{C}_0)$ makes sense not only for ordinary categories, but also for homotopy theories. In other words, suppose that $\mathcal{C}_0$ is a category enriched over topological spaces (so that for every pair of objects $X, Y \in \mathcal{C}_0$, we have a mapping space $\mathrm{Map}_{\mathcal{C}_0}(X, Y)$). Then we can define a new topological category $\mathrm{Pro}(\mathcal{C}_0)$. Roughly speaking, the objects of $\mathrm{Pro}(\mathcal{C}_0)$ are given by formal filtered limits "$\varprojlim C_\alpha$" in $\mathcal{C}_0$, and the morphisms are described by the formula

$$\mathrm{Map}(\text{“}\varprojlim C_\alpha\text{”}, \text{“}\varprojlim D_\beta\text{”}) = \mathrm{holim}_\beta \,\mathrm{hocolim}_\alpha \mathrm{Map}(C_\alpha, D_\beta).$$

To really make this idea precise requires the machinery of higher category theory; we will be content to work with this construction in an informal way.

We now specialize this construction to the case of interest. Let $\mathfrak{S}$ denote the category of spaces, $\mathfrak{S}_p$ the category of p-finite spaces, and $\mathfrak{S}_p^\vee$ the category $\mathrm{Pro}(\mathfrak{S}_p)$ of pro-objects in $\mathfrak{S}_p$. We will refer to $\mathfrak{S}_p^\vee$ as the category of *p-profinite spaces*.

There is a canonical functor $G : \mathfrak{S}_p^\vee \to \mathfrak{S}$, which carries a formal inverse limit "$\varprojlim C_\alpha$" to the space $\mathrm{holim}\, C_\alpha$. If we restrict to a suitable subcategory of $\mathfrak{S}_p^\vee$ by imposing finiteness and connectivity conditions, then the functor G is fully faithful; its essential image being (a suitable subcategory of) the category of p-complete spaces. We will discuss this point in more detail in a future lecture.

The functor G has a left adjoint $X \mapsto X^\vee$, which we will refer to as the functor of *p-profinite completion*. The functor $^\vee$ carries a topological space X to the formal inverse limit $X^\vee = \text{“}\varprojlim X_\alpha\text{”}$, where X_α ranges over all p-finite spaces equipped with a map to X. If X is itself p-finite, then we can identify this inverse limit with X itself.

Definition 6. Let X be a p-profinite space. We let $\mathrm{H}^n(X) = \mathrm{H}^n(X; \mathbf{F}_p)$ denote the set of homotopy classes of maps from X into an Eilenberg-MacLane space $K(\mathbf{F}_p, n)$ in the p-profinite category $\mathfrak{S}_p^\vee$.

Since $K(\mathbf{F}_p, n)$ is p-finite, we see that

$$\mathrm{H}^n(\text{“}\varprojlim X_\alpha\text{”}) \simeq \varinjlim \mathrm{H}^n(X_\alpha).$$

It follows that for any p-profinite space X, the cohomology $\mathrm{H}^*(X) \simeq \oplus_n \mathrm{H}^n(X)$ is a filtere colimit of the cohomology rings of a collection of p-finite spaces, and therefore inherits the structure of an unstable algebra over the Steenrod algebra.

Remark 7. If X is a topological space, then the cohomology $\mathrm{H}^*(X; \mathbf{F}_p)$ (in the usual sense) can be identified with the cohomology $\mathrm{H}^*(X^\vee)$ of the p-profinite completion of X, defined as in Definition 6.

The process of extracting cohomology does *not* generally commute with the inverse limit functor $G : \mathfrak{S}_p^\vee \to \mathfrak{S}$, unless we make suitable finiteness assumptions.

We now discuss the existence of mapping objects in the p-profinite category.

Proposition 8. *Let X be a p-profinite space, and let V be a finite dimensional vector space over $\mathbf{F}_p$. Then there exists a p-profinite space X^{BV} equipped with an evaluation map $X^{BV} \times BV \to X$ with the following universal property: for any p-profinite space Y, the induced map*

$$\theta : \mathrm{Map}(Y, X^{BV}) \to \mathrm{Map}(Y \times BV, X)$$

is a weak homotopy equivalence.

Proof. If $X = \text{``}\varprojlim X_\alpha\text{''}$, then we can take $X^{BV} = \text{``}\varprojlim X_\alpha^{BV}\text{''}$ (here we are using the fact that each X_α^{BV} is again p-finite). We claim tht X^{BV} has the appropriate universal property. For any p-profinite space Y, we can identify θ with a map

$$\operatorname{holim} \operatorname{Map}(Y, X_\alpha^{BV}) \simeq \operatorname{Map}(Y, X^{BV}) \to \operatorname{Map}(Y \times BV, X) \simeq \operatorname{holim} \operatorname{Map}(Y \times BV, X_\alpha).$$

It will therefore suffice to prove the result after replacing X by X_α, so we may assume that X is p-finite. Let $Y = \text{``}\varprojlim Y_\beta\text{''}$. Then the map θ can be identified with

$$\operatorname{hocolim} \operatorname{Map}(Y_\beta, X^{BV}) \simeq \operatorname{Map}(Y, X^{BV}) \to \operatorname{Map}(Y \times BV, X) \simeq \operatorname{hocolim} \operatorname{Map}(Y_\beta \times BV, X),$$

where the last equivalence follows from the observation that

$$Y \times BV \simeq \text{``}\varprojlim Y_\beta \times BV\text{''}$$

is a product for Y and BV in the p-profinite category. We may therefore assume that Y is p-finite as well, in which case the result is obvious. □

Remark 9. Proposition 9 remains valid if we replace BV by an arbitrary p-finite space. However, it is not valid if BV is a general p-profinite space; the p-profinite category $\mathfrak{S}_p^\vee$ does not have internal mapping objects in general.

Remark 10. Let $X = \text{``}\varprojlim X_\alpha\text{''}$ and $Y = \text{``}\varprojlim Y_\beta\text{''}$ be p-profinite spaces. Then $\text{``}\varprojlim X_\alpha \times Y_\beta\text{''}$ is a product for X and Y in the category of p-profinite spaces. Applying the Kunneth theorem to the p-finite spaces X_α and Y_β, we deduce

$$\mathrm{H}^*(X \times Y) \simeq \varinjlim \mathrm{H}^*(X_\alpha \times Y_\beta) \simeq \varinjlim \mathrm{H}^* X_\alpha \otimes \mathrm{H}^* Y_\beta \simeq \mathrm{H}^* X \otimes \mathrm{H}^* Y.$$

Let us now assume that $p = 2$. Let X be a p-profinite space. The evaluation map $X^{BV} \times BV \to X$ induces a map on cohomology

$$\mathrm{H}^* X \to \mathrm{H}^*(X^{BV} \times BV) \simeq \mathrm{H}^*(X^{BV}) \otimes \mathrm{H}^*(BV),$$

which is adjoint to a map $\psi : T_V \mathrm{H}^*(X) \to \mathrm{H}^*(X^{BV})$.

Theorem 11. *The map ψ is an isomorphism, for every 2-profinite space X.*

Proof. The proof when X is 2-finite was given in the previous lecture. In general, write $X = \text{``}\varprojlim X_\alpha\text{''}$. Then we have

$$\begin{aligned} T_V \mathrm{H}^*(X) &\simeq T_V \varinjlim \mathrm{H}^*(X_\alpha) \\ &\simeq \varinjlim T_V \mathrm{H}^*(X_\alpha) \\ &\simeq \varinjlim \mathrm{H}^*(X_\alpha^{BV}) \\ &\simeq \mathrm{H}^*(X^{BV}). \end{aligned}$$

□

Using this result, we get a measure of exactly how the ψ might fail to be an isomorphism when we work in the usual category of spaces. For any space X, we have

$$T_V \mathrm{H}^*(X) \simeq T_V \mathrm{H}^*(X^\vee) \simeq \mathrm{H}^*(X^\vee)^{BV} \to \mathrm{H}^*(X^{BV})^\vee.$$

In other words, the failure of T_V to compute the cohomology of mapping spaces is measured by the failure of the formation of mapping spaces to commute with profinite completion.