18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

p-adic Homotopy Theory (Lecture 27)

In this lecture we will continue to study the category $\mathfrak{S}_p^\vee$ of p-profinite spaces, where p is a prime number. Our main goal is to connect $\mathfrak{S}_p^\vee$ with the category of E_∞-algebras over the field $\overline{\mathbf{F}}_p$, following the ideas of Dwyer, Hopkins, and Mandell.

We begin with a brief review of rational homotopy theory. For any topological space X, Sullivan showed how to construct a model for the rational cochain complex $C^*(X;\mathbf{Q})$ which admits the structure of a *differential graded algebra* over $\mathbf{Q}$. The work of Quillen and Sullivan shows that the differential graded algebra $C^*(X;\mathbf{Q})$ completely encodes the "rational" structure of the space X. For example, if X is a simply connected space whose homology groups $\mathrm{H}_i(X;\mathbf{Z})$ are finitely generated, then the space $X_{\mathbf{Q}} = \mathrm{Map}(C^*(X;\mathbf{Q}), \mathbf{Q})$ is a *rationalization* of X: that is, there is a map $X \to X_{\mathbf{Q}}$ which induces an isomorphism on rational homology. Here the mapping space $\mathrm{Map}(C^*(X;\mathbf{Q}), \mathbf{Q})$ is computed in the homotopy theory of differential graded algebras over $\mathbf{Q}$.

Our goal is to establish an analogue of this result, where we replace the field $\mathbf{Q}$ by a field $\mathbf{F}_p$ of characteristic p. In this case, we cannot generally choose a model for $C^*(X;\mathbf{F}_p)$ by a differential graded algebra (this is the origin of the existence of Steenrod operations). However, we can still view $C^*(X;\mathbf{F}_p)$ as an E_∞-algebra, and ask to what extent this E_∞-algebra determines the homotopy type of X. We first observe that $C^*(X;\mathbf{F}_p)$ depends only on the p-profinite completion of X. For *any* p-profinite space $Y =$ "$\varprojlim Y_\alpha$", we can define $C^*(Y;\mathbf{F}_p) = \varinjlim C^*(Y_\alpha;\mathbf{F}_p)$. If Y is the p-profinite completion of a topological space X, then the canonical maps $X \to Y_\alpha$ induce a map of E_∞-algebras

$$\theta : C^*(Y;\mathbf{F}_p) \simeq \varinjlim C^*(Y_\alpha;\mathbf{F}_p) \to C^*(X;\mathbf{F}_p).$$

Since the the Eilenberg-MacLane spaces $K(\mathbf{F}_p, n)$ are p-finite and represent the functor $X \mapsto \mathrm{H}^n(X;\mathbf{F}_p)$, we deduce that θ is an isomorphism on cohomology.

Let k be *any* field of characteristic p. Then, for every p-profinite space $Y =$ "$\varprojlim Y_\alpha$", we define

$$C^*(Y;k) = C^*(Y;\mathbf{F}_p) \otimes_{\mathbf{F}_p} k \simeq \varinjlim C^*(Y_\alpha;k).$$

Warning 1. If Y is the p-profinite completion of a space X, then we again have a canonical map of E_∞-algebras

$$C^*(Y;k) \to C^*(X;k),$$

but this map is generally *not* an isomorphism on cohomology, since the Eilenberg-MacLane spaces $K(k,n)$ are generally not p-finite.

Our goal is to prove the following:

Theorem 2. *Let k be an algebraically closed field of characteristic p. The functor*

$$X \mapsto C^*(X;k)$$

induces a fully faithful embedding from the homotopy theory of p-profinite spaces to the homotopy theory of E_∞-algebras over k.

We first need the following lemma:

Lemma 3. *The functor F defined by the formula*

$$X \mapsto C^*(X;k)$$

carries homotopy limits of p-profinite spaces to homotopy colimits of E_∞-algebras over k.

Proof. By general nonsense, it will suffice to prove that F carries filtered limits to filtered colimits and finite limits to finite colimits.

For any category $\mathcal{C}$, the category $\mathrm{Pro}(\mathcal{C})$ can be characterized by the following universal property: it is freely generated by $\mathcal{C}$ under filtered limits. In other words, $\mathrm{Pro}(\mathcal{C})$ admits filtered limits, and if $\mathcal{D}$ is any other category which admits filtered limits, then functors from $\mathcal{C}$ to $\mathcal{D}$ extend uniquely (up to equivalence) to functors from $\mathrm{Pro}(\mathcal{C})$ to $\mathcal{D}$ which preserve filtered limits. By construction, the functor F is the unique extension of the functor $X \mapsto C^*(X;\mathbf{F}_p)$ on p-finite spaces which carries filtered limits to filtered colimits.

To show that F preserves finite limits to finite colimits, it will suffice to show that F carries final objects to initial objects, and homotopy pullback diagrams to homotopy pushout diagrams. The first assertion is evident: $F(*) \simeq k$ is the initial E_∞-algebra over k. To handle the case of pullbacks, we note that every homotopy pullback square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

of p-profinite spaces is a filtered limit of homotopy pullback squares between p-finite spaces. We may therefore assume that the diagram consists of p-finite spaces, in which case we proved earlier that the diagram

$$\begin{array}{ccc} C^*(X';\mathbf{F}_p) & \longleftarrow & C^*(X;\mathbf{F}_p) \\ \uparrow & & \uparrow \\ C^*(Y';\mathbf{F}_p) & \longleftarrow & C^*(Y;\mathbf{F}_p) \end{array}$$

is a homotopy pushout square of E_∞-algebras over $\mathbf{F}_p$. The desired result now follows by tensoring over $\mathbf{F}_p$ with k. □

Lemma 4. *Let $\mathcal{K}$ be a collection of p-profinite spaces. Suppose that $\mathcal{K}$ contains every Eilenberg-MacLane space $K(\mathbf{F}_p, n)$ and is closed under the formation of homotopy limits. Then $\mathcal{K}$ contains all p-profinite spaces X.*

Proof. Every p-profinite space X is a filtered homotopy limit of p-finite spaces. We may therefore assume that X is finite. In this case, X admits a finite filtration

$$X \simeq X_m \to X_{m-1} \to \ldots \to X_0 \simeq *$$

where, for each i, we have a homotopy pullback diagram

$$\begin{array}{ccc} X_{i+1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & K(\mathbf{F}_p, n_i). \end{array}$$

It follows by induction on i that each X_i belongs to $\mathcal{K}$. □

We now turn to the proof of Theorem 2. Fix a p-profinite space Y. For every p-profinite space X, we have a canonical map

$$\theta_X : \mathrm{Map}(Y, X) \to \mathrm{Map}_k(C^*(X; k), C^*(Y; k)).$$

Let $\mathcal{K}$ denote the collection of all p-profinite spaces X for which θ_X is a homotopy equivalence. Lemma 3 implies that both sides above are compatible with the formation of homotopy limits in X, so $\mathcal{K}$ is closed under the formation of homotopy limits. It will therefore suffice to show that every Eilenberg-MacLane space $K(\mathbf{F}_p, n)$ belongs to $\mathcal{K}$.

For each i, the map $\theta_{K(\mathbf{F}_p,n)}$ induces a map

$$\mathrm{H}^{n-i}(Y; \mathbf{F}_p) \simeq \pi_i \mathrm{Map}(Y, K(\mathbf{F}_p, n)) \to \pi_i \mathrm{Map}_k(C^*(K(\mathbf{F}_p, n); k), C^*(Y; k)) \simeq \pi_i \mathrm{Map}_{\mathbf{F}_p}(C^*(K(\mathbf{F}_p, n); \mathbf{F}_p), C^*(Y; k));$$

we wish to show that these maps are isomorphisms.

We now specialize to the case $p = 2$, where we have described the cochain complex $C^*(K(\mathbf{F}_p, n); \mathbf{F}_p)$ as an E_∞-algebra over $\mathbf{F}_p$: namely, we have a pushout diagram of E_∞-algebras

$$\begin{array}{ccc} \mathcal{F}(n) & \xrightarrow{u} & \mathcal{F}(n) \\ \downarrow & & \downarrow \\ \mathbf{F}_p & \longrightarrow & C^*(K(\mathbf{F}_p, n); \mathbf{F}_p) \end{array}$$

where the map u classifies the cohomology operation $\mathrm{id} - \mathrm{Sq}^0$. It follows that we have a long exact sequence of homotopy groups

$$\ldots \to \mathrm{H}^{n-i-1}(Y; k) \to \pi_i \mathrm{Map}_{\mathbf{F}_p}(C^*(K(\mathbf{F}_p, n); \mathbf{F}_p), C^*(Y; k)) \to \mathrm{H}^{n-i}(Y; k) \overset{\mathrm{id}-\mathrm{Sq}^0}{\to} \mathrm{H}^{n-i}(Y; k) \to \ldots$$

To compute the homotopy groups of $\mathrm{Map}_{\mathbf{F}_p}(C^*(K(\mathbf{F}_p, n); \mathbf{F}_p), C^*(Y; k))$, we need to understand the cohomology ring $\mathrm{H}^*(Y; k)$ as an algebra over the big Steenrod algebra $\mathcal{A}^{\mathrm{Big}}$. We observe that

$$\mathrm{H}^*(Y; k) \simeq \mathrm{H}^*(Y; \mathbf{F}_p) \otimes_{\mathbf{F}_p} k.$$

The operation Sq^0 acts by the identity on the first factor, and by the Frobenius map $x \mapsto x^p$ on the field k. Since k is algebraically closed, we have an Artin-Schreier sequence

$$0 \to \mathbf{F}_p \to k \overset{v}{\to} k \to 0$$

where v is given by $v(x) = x - x^p$. It follows that the operation $\mathrm{id} - \mathrm{Sq}^0$ on $\mathrm{H}^*(Y; k)$ is surjective, with kernel $\mathrm{H}^*(Y; \mathbf{F}_p)$. Thus the long exact sequence above yields a sequence of isomorphisms

$$\pi_i \mathrm{Map}_{\mathbf{F}_p}(C^*(K(\mathbf{F}_p, n); \mathbf{F}_p) C^*(Y; k)) \simeq \mathrm{H}^{n-i}(Y; \mathbf{F}_p)$$

as desired.

Remark 5. The proof of Theorem 2 does not require that k is algebraically closed, only that k admits no Artin-Schreier extensions (that is, that any equation $x - x^p = \lambda$ admits a solution in k). Equivalently, it requires that the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ have vanishing mod-p cohomology.

Remark 6. Theorem 2 is false for a general field k of characteristic p; for example, it fails when $k = \mathbf{F}_p$. However, we can obtain a more general statement as follows. Suppose that X is a p-profinite sheaf of spaces on the étale topos of $\mathrm{Spec}\, k$; in other words, that X is a p-profinite space equipped with a suitably continuous action σ of the Galois group $\mathrm{Gal}(\overline{k}/k)$. In this case, we get a Galois action on the cochain complex

$$C^*(X; \overline{k}).$$

Using descent theory, we can extract from this an E_∞-algebra of Galois invariants $C^*_\sigma(X;k)$, which we can regard as a σ-twisted version of the usual cochain complex $C^*(X;k)$ (these cochain complexes can be identified in the case where the action of σ is trivial). The construction

$$(X, \sigma) \mapsto C^*_\sigma(X;k)$$

determines a functor from p-profinite sheaves on $\operatorname{Spec} k$ to the category of E_∞-algebras over k, and *this* functor is again fully faithful.