18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Atomicity (Lecture 28)

Let V be a finite dimensional vector space over $\mathbf{F}_2$, and let T_V denote Lannes's T-functor. In previous lectures we have established two very important properties of T_V:

- The functor T_V is exact.
- For every 2-profinite space X, there is a canonical isomorphism

$$T_V \,\mathrm{H}^* X \simeq \mathrm{H}^* X^{BV}.$$

Our goal in this lecture is to deduce a conceptual consequence of these facts, which makes no mention of modules over the Steenrod algebra.

Definition 1. Let $\mathcal{C}$ be a (topological) category which admits finite (homotopy) limits and colimits. We will say that an object $K \in \mathcal{C}$ is *atomic* if the following conditions are satisfied:

(a) For every $X \in \mathcal{C}$, there exists an object $X^K \in \mathcal{C}$ and an evaluation map $e : X^K \times K \to X$ with the following universal property: for every $Y \in \mathcal{C}$, composition with e induces a homotopy equivalence

$$\mathrm{Map}(Y, X^K) \to \mathrm{Map}(Y \times K, X).$$

(b) The functor $X \mapsto X^K$ preserves finite (homotopy) colimits.

Example 2. Let $\mathcal{C}$ be the category of spaces. Then the point $K = *$ is an atomic object of $\mathcal{C}$.

We will be primarily interested in the case where $\mathcal{C} = \mathfrak{S}_p^\vee$ is the category of p-profinite spaces. We note that $\mathcal{C}$ admits homotopy colimits. This is perhaps not completely obvious, since the collection of p-finite spaces is not closed under homotopy colimits. For example, given a diagram of p-finite spaces

$$X \leftarrow Y \to X',$$

the (homotopy) pushout of this diagram in $\mathfrak{S}_p^\vee$ is obtained as the p-profinite completion of the analogous homotopy pushout $X \coprod_Y X'$ in the category of spaces.

Suppose that K is a p-finite space; we wish to study the condition that K be atomic. Condition (a) is automatic. Condition (b) can be divided into two assertions:

(b_0) The functor $X \mapsto X^K$ preserves initial objects. This is true if and only if K is nonempty.

(b_1) The functor $X \mapsto X^K$ preseves homotopy pushouts.

Condition (b_1) implies, for example, that for every pair of p-profinite spaces X and Y, we have $(X \coprod Y)^K \simeq X^K \coprod Y^K$; in other words, every map from K to a disjoint union must factor through one of the summands. This is equivalent to the assertion that K is connected. A priori, the condition of atomicity is much stronger: it implies, for example, that K cannot be written nontrivially as a homotopy pushout of p-profinite spaces. Nevertheless, we have the following result:

Theorem 3. *Let K be a connected p-finite space. Then K is an atomic object of the p-profinite category.*

We will prove Theorem 3 in the next lecture. For now, we will be content to study the special case where $K = BV$, where V is a finite dimensional vector space over $\mathbf{F}_p$ (and the prime p is equal to 2). In this case, we need to show:

Proposition 4. *Let V be a finite dimensional vector space over $\mathbf{F}_p$, and let*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a homotopy pushout diagram of p-profinite spaces. Then the induced diagram

$$\begin{array}{ccc} X^{BV} & \longrightarrow & X'^{BV} \\ \downarrow & & \downarrow \\ Y^{BV} & \longrightarrow & Y'^{BV} \end{array}$$

is also a homotopy pushout diagram.

Remark 5. Let $f : X \to Y$ be a map of p-profinite spaces. Then f is an equivalence if and only if induces an isomorphism $\mathrm{H}^*(Y) \to \mathrm{H}^*(X)$. The "only if" direction is obvious. For the converse, let us suppose that f induces an isomorphism of cohomology. We will show that f induces a weak homotopy equivalence

$$\phi_Z : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

for every p-profinite space Z. We may immediately reduce to the case where Z is p-finite (since the class of weak homotopy equivalences is stable under homotopy limits). In this case, we have a finite filtration

$$Z \simeq Z_m \to Z_{m-1} \to \ldots \to Z_0 \simeq *$$

by principal fibrations with fiber $K(\mathbf{F}_p, n_i)$; we will show that ϕ_{Z_i} is a weak homotopy equivalence using induction on i. We have a homotopy pullback diagram

$$\begin{array}{ccc} Z_{i+1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & K(\mathbf{F}_p, n_i + 1). \end{array}$$

Consequently, to show that $\phi_{Z_{i+1}}$ is a homotopy equivalence, it will suffice to show that ϕ_*, ϕ_{Z_i}, and $\phi_{K(\mathbf{F}_p, n_i+1)}$ are weak homotopy equivalences. The first claim is obvious, the second follows from the inductive hypothesis, and the third follows from our hypothesis on f since

$$\pi_k \,\mathrm{Map}(Y, K(\mathbf{F}_p, n_i + 1)) \simeq \mathrm{H}^{n_i+1-k}(Y) \simeq \mathrm{H}^{n_i+1-k}(X) \simeq \pi_k \,\mathrm{Map}(X, K(\mathbf{F}_p, n_i + 1)).$$

Proof of Proposition 4. Let Z denote a homotopy pushout of Y^{BV} and X'^{BV} over X^{BV}. The evaluation maps $Y^{BV} \times BV \to Y$ and $X'^{BV} \times BV \to X'$ glue together to give a map $Z \times BV \to Y'$. We therefore have a map of homotopy pushout diagrams

$$\begin{array}{ccc} X^{BV} \times BV & \longrightarrow & X'^{BV} \times BV \\ \downarrow & & \downarrow \\ Y^{BV} \times BV & \longrightarrow & Z \times BV \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y', \end{array}$$

which induces a map of long exact sequences

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{H}^{*-1} X & \longrightarrow & \mathrm{H}^* Y' & \longrightarrow & \mathrm{H}^* Y \oplus \mathrm{H}^* X' & \longrightarrow & \mathrm{H}^* X & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \mathrm{H}^* X^{BV} \otimes \mathrm{H}^* X & \longrightarrow & \mathrm{H}^* Z \otimes \mathrm{H}^* BV & \longrightarrow & (\mathrm{H}^* Y^{BV} \oplus \mathrm{H}^* X'^{BV}) \otimes \mathrm{H}^* BV & \longrightarrow & \mathrm{H}^* X^{BV} \otimes \mathrm{H}^* BV & \longrightarrow
\end{array}$$

Since T_V is exact, this diagram is adjoint to another map of long exact sequences

$$\begin{array}{ccccccccc}
\longrightarrow & T_V \mathrm{H}^{*-1} X & \longrightarrow & T_V \mathrm{H}^* Y' & \longrightarrow & T_V \mathrm{H}^* Y \oplus T_V \mathrm{H}^* X' & \longrightarrow & T_V \mathrm{H}^* X & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \mathrm{H}^{*-1} X^{BV} & \longrightarrow & \mathrm{H}^* Z & \longrightarrow & \mathrm{H}^* Y^{BV} \oplus \mathrm{H}^* X'^{BV} & \longrightarrow & \mathrm{H}^* X^{BV} & \longrightarrow
\end{array}$$

Using the five-lemma, we deduce that the map $T_V \mathrm{H}^* Y' \to \mathrm{H}^* Z$ is an isomorphism. This map fits into a commutative diagram

$$\begin{array}{ccc}
T_V \mathrm{H}^* Y' & \longrightarrow & \mathrm{H}^* Y'^{BV} \\
& \searrow \quad \swarrow_{\alpha} & \\
& \mathrm{H}^* Z, &
\end{array}$$

where α is induced by the map of p-profinite space $f : Z \to Y'^{BV}$. Using the two-out-of-three property, we deduce that α is an isomorphism. It follows from Remark 5 that f is an equivalence of p-profinite spaces, as desired. □

We now wish to prove the atomicity of a larger class of p-finite spaces. First, we reformulate the definition of atomicity. First, we introduce a bit of notation. For every p-finite space K, we let $\mathfrak{S}^{\vee}_{p,/K}$ denote the category of *p-profinite spaces over K*, so that an object of $\mathfrak{S}^{\vee}_{p,/K}$ is a map $X \to K$ in the p-profinite category. Given a map $q : K \to K'$, we have a pullback functor $q^* : \mathfrak{S}^{\vee}_{p,/K'} \to \mathfrak{S}^{\vee}_{p,/K}$, which is given by forming the homotopy pullback

$$X \mapsto X \times_K K'.$$

This functor has a right adjoint, which we will denote by q_*. In the case where K' is a point, q_* assigns to a map $f : X \to K$ the p-profinite space of sections of f (more precisely, $q_* X$ has the following universal property: for every p-profinite space Y, we have

$$\mathrm{Map}(Y, q_* X) \simeq \mathrm{Map}(Y \times K, X) \times_{\mathrm{Map}(Y \times K, K)} \{\pi_2\},$$

where π_2 denotes the projection onto the second factor. In particular, if X is a product $X_0 \times K$, then $q_* X$ is equivalent to the mapping space X_0^K.

Proposition 6. *Let K be a p-finite space. The following conditions are equivalent:*

(1) *K is an atomic object of the p-profinite category.*

(2) *Let $q : K \to *$ denote the projection. Then the functor $q_* : \mathfrak{S}^{\vee}_{p,/K} \to \mathfrak{S}^{\vee}_{p}$ preserves finite homotopy colimits.*

Proof. By definition, K is atomic if and only if the composite functor $q_* q^*$ preserves finite homotopy colimits. Since q^* preserves finite homotopy colimits (being a left adjoint), the implication (2) ⇒ (1) is obvious. For the converse, we observe that we have a natural equivalence

$$q_* X \simeq X^K \times_{K^K} \{\mathrm{id}_K\},$$

and the functor $Y \mapsto Y \times_{K^K} \{\mathrm{id}_K\}$ preserves all homotopy colimits. □

Corollary 7. *Suppose given a fiber sequence*

$$F \xrightarrow{f} E \xrightarrow{g} B$$

of connected p-finite spaces. If F and B are atomic (when regarded as p-profinite spaces), then E is atomic (when regarded as a p-profinite space).

Proof. Let q denote the projection from B to a point. We wish to show that the functor $(q \circ g)_* = q_* \circ g_*$ preserves finite homotopy colimits. Since B is atomic, q_* preserves finite homotopy colimits. It will therefore suffice to show that g_* preserves finite homotopy colimits. For this, it suffices to show that $i^* g_*$ preserves finite homotopy colimits, where i denotes the inclusion of any point b into B. We have an equivalence

$$i^* g_* \simeq g'_* f^*,$$

where g' denotes the projection $E \times_B \{b\} \simeq F \to \{b\}$. The functor f^* preserves all homotopy colimits (since it is a left adjoint), and g'_* preserves finite homotopy colimits since F is assumed to be atomic. $\square$

Corollary 8. *Let G be a finite p-group. Then the classifying space BG is an atomic object in the p-profinite category.*