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18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

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Atomicity of Connected p -Finite Spaces (Lecture 29)

In this lecture, we will show that every connected p -finite space K is an atomic object of the category of p -profinite spaces. We begin with the following result, which is a soft version of the convergence of the homology spectral sequence for a cosimplicial space:

Proposition 1. *Consider the functor F :*

$$X \mapsto C^*X$$

from the category of p -profinite spaces \mathfrak{S}_p^\vee to the category of complexes of \mathbf{F}_p -vector spaces. Then F carries totalizations (of cosimplicial p -profinite spaces) to geometric realizations (of simplicial objects in the category of complexes).

Proof. The functor F factors as a composition $F'' \circ F'$, where:

- The functor F' is given by the formula

$$X \mapsto C^*X,$$

but C^*X is regarded as an E_∞ -algebra over \mathbf{F}_p .

- The functor F'' is the forgetful functor from E_∞ -algebras over \mathbf{F}_p to complexes of \mathbf{F}_p -vector spaces.

We have shown that the functor F' carries arbitrary homotopy limits of p -profinite spaces to homotopy colimits of E_∞ -algebras; in particular, it carries totalizations to geometric realizations. It now suffices to observe that F'' preserves geometric realizations of simplicial objects. \square

Remark 2. In concrete terms, the statement that F'' preserves geometric realizations amounts to the following observation. Let A_\bullet be a simplicial object in the category of E_∞ -algebras over \mathbf{F}_p , and let $|A_\bullet|$ be the geometric realization of A_\bullet as a complex of \mathbf{F}_p -vector spaces. Then $|A_\bullet|$ inherits the structure of an E_∞ -algebra. For example, the multiplication on $|A_\bullet|$ arises in the following way: the tensor product $|A_\bullet| \otimes |A_\bullet|$ can be identified with the homotopy colimit of the bisimplicial complex $B_{\bullet,\bullet}$ given by the formula $B_{m,n} = A_m \otimes A_n$. This homotopy colimit can be computed as the geometric realization of the diagonal simplicial object $B_\bullet : [n] \mapsto B_{n,n}$, and we have a map

$$|B_\bullet| \rightarrow |A_\bullet|$$

which is induced by the maps $B_n = B_{n,n} = A_n \otimes A_n \rightarrow A_n$ given by the multiplication on A_n .

Theorem 3. *Let \mathfrak{S}_p^\vee denote the category of p -profinite spaces, and $c\mathfrak{S}_p^\vee$ the category of cosimplicial p -profinite spaces. Then the totalization functor*

$$c\mathfrak{S}_p^\vee \rightarrow \mathfrak{S}_p^\vee$$

commutes with homotopy pushouts.

Proof. Suppose given a homotopy pushout diagram

$$\begin{array}{ccc} X^\bullet & \longrightarrow & X'^\bullet \\ \downarrow & & \downarrow \\ Y^\bullet & \longrightarrow & Y'^\bullet \end{array}$$

of p -profinite spaces. We wish to show that the associated diagram

$$\begin{array}{ccc} \mathrm{Tot}(X^\bullet) & \longrightarrow & \mathrm{Tot}(X'^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Tot}(Y^\bullet) & \longrightarrow & \mathrm{Tot}(Y'^\bullet) \end{array}$$

is again a homotopy pushout diagram. In other words, we wish to show that the canonical map

$$\mathrm{Tot}(X'^\bullet) \coprod_{\mathrm{Tot}(X^\bullet)} \mathrm{Tot}(Y^\bullet) \rightarrow \mathrm{Tot}(Y'^\bullet)$$

is an equivalence of p -profinite spaces. As we saw in the last lecture, it will suffice to show that this map induces an isomorphism on cohomology. In other words, it suffices to show that the induced map

$$C^*(\mathrm{Tot}(Y'^\bullet)) \rightarrow C^*(\mathrm{Tot}(X'^\bullet) \coprod_{\mathrm{Tot}(X^\bullet)} \mathrm{Tot}(Y^\bullet))$$

is a quasi-isomorphism. By excision, the right hand side can be identified with the homotopy fiber product

$$(C^* \mathrm{Tot} X'^\bullet) \times_{C^* \mathrm{Tot} X^\bullet} (C^* \mathrm{Tot} Y^\bullet).$$

In other words, we must show that the diagram

$$\begin{array}{ccc} C^* \mathrm{Tot}(X^\bullet) & \longleftarrow & C^* \mathrm{Tot}(X'^\bullet) \\ \uparrow & & \uparrow \\ C^* \mathrm{Tot}(Y^\bullet) & \longleftarrow & C^* \mathrm{Tot}(Y'^\bullet) \end{array}$$

is a homotopy pullback square (in the category of complexes of \mathbf{F}_p -vector spaces). Using Proposition 1 we can rewrite this square as

$$\begin{array}{ccc} |C^* X^\bullet| & \longleftarrow & |C^* X'^\bullet| \\ \uparrow & & \uparrow \\ |C^* Y^\bullet| & \longleftarrow & |C^* Y'^\bullet|. \end{array}$$

The homotopy theory of complexes of \mathbf{F}_p -vector spaces is *stable*: that is, homotopy pullback squares are the same as homotopy pushout squares. It will therefore suffice to show that the diagram above is a homotopy pushout square. Since the collection of homotopy pushout squares is stable under homotopy colimits (and in particular under geometric realizations), we are reduced to showing that each diagram

$$\begin{array}{ccc} C^* X^n & \longleftarrow & C^* X'^n \\ \uparrow & & \uparrow \\ C^* Y^n & \longleftarrow & C^* Y'^n \end{array}$$

is a homotopy pushout square. Using stability again, we need only show that this diagram is a homotopy pullback square. This follows by excision from the assumption that

$$\begin{array}{ccc} X^n & \longrightarrow & X'^n \\ \downarrow & & \downarrow \\ Y^n & \longrightarrow & Y'^n \end{array}$$

is a homotopy pushout square of p -profinite spaces. \square

Remark 4. The same argument can be used to show that the formation of filtered limits of p -profinite spaces preserves homotopy pushout squares.

Corollary 5. *Let K_\bullet be a simplicial object in the category of p -finite spaces. Suppose that each K_n is atomic in the p -profinite category, and that the geometric realization $K = |K_\bullet|$ is again p -finite. Then K is also atomic in the p -profinite category.*

Proof. Let X be any p -profinite space. Since each K_n is atomic, we can construct a mapping object X^{K_n} . The universal property of X^{K_n} shows that it depends on K_n in a contravariantly functorial manner, so that $F(X) = X^{K_\bullet}$ is a cosimplicial object in the category of p -profinite spaces, and the totalization of this cosimplicial object can be identified with X^K . Consequently, the functor

$$X \mapsto X^K$$

can be factored as a composition

$$\mathfrak{S}_p^\vee \xrightarrow{F} c\mathfrak{S}_p^\vee \xrightarrow{\text{Tot}} \mathfrak{S}_p^\vee.$$

Since each K_n is atomic, the functor F preserves homotopy pushouts. Theorem 3 implies that Tot preserves homotopy pushouts as well, so the functor $X \mapsto X^K$ preserves homotopy pushouts. Since K is clearly nonempty, we deduce that K is atomic. \square

Corollary 6. *Let G be a finite p -group and $n \geq 1$ an integer; assume that G is abelian if $n > 1$. Then the Eilenberg-MacLane space $K(G, n)$ is an atomic object of the p -profinite category.*

Proof. The case $n = 1$ was handled in the previous lecture, using Lannes' T-functor. The proof in general goes by induction on n . Choose a fibration $E_0 \rightarrow K(G, n)$, where E_0 is contractible, and consider the associated simplicial object defined by the formula

$$E_k = E_0 \times_{K(G, n)} E_0 \times \dots \times_{K(G, n)} E_0$$

(where the factor E_0 appears $(k + 1)$ -times). Then E_\bullet is a simplicial object whose geometric realization $|E_\bullet|$ can be identified with $K(G, n)$. Moreover, each E_k is homotopy equivalent to the Eilenberg-MacLane space $K(G^k, n - 1)$. The inductive hypothesis implies that each E_k is atomic. Using Corollary 5, we conclude that $|E_\bullet| \simeq K(G, n)$ is atomic as well. \square

Theorem 7. *Let X be a connected p -finite space. Then X is an atomic object of the p -profinite category.*

Proof. The space X admits a Postnikov tower

$$X \simeq X_m \rightarrow X_{m-1} \rightarrow \dots \rightarrow X_0 \simeq *,$$

where

$$\pi_k X_i = \begin{cases} \pi_k X & \text{if } k \leq i \\ 0 & \text{otherwise.} \end{cases}$$

We will show by induction that each X_i is atomic. For each i , we have a fiber sequence

$$K(\pi_i X, i) \rightarrow X_i \rightarrow X_{i-1}$$

of connected p -finite spaces. Consequently, to show that X_i is atomic, it will suffice to show that X_{i-1} and $K(\pi_i X, i)$ are atomic. The first follows from the inductive hypothesis and the second from Corollary 6. \square

We conclude this lecture with a complement to Theorem 7, which indicates the strength of the atomicity condition.

Proposition 8. *Let X be an atomic object in the category \mathfrak{S} of spaces. Then X is (weakly) contractible.*

Proof. By assumption $\text{Map}(X, \bullet)$ commutes with finite colimits. In particular, $\text{Map}(X, \emptyset)$ is empty; this implies that X is nonempty. Since $\text{Map}(X, \bullet)$ commutes with disjoint unions, we conclude that X is connected. Suppose (for a contradiction) that X is not weakly contractible.

Without loss of generality, we can assume that X is a CW complex. It can therefore be written as the homotopy colimit of its skeleta

$$\text{sk}^0 X \rightarrow \text{sk}^1 X \rightarrow \text{sk}^2 X \rightarrow \dots$$

Let X' denote the mapping telescope for this sequence of maps, so that we have a canonical homotopy equivalence $X' \simeq X$. The telescope X' admits a decomposition

$$X' \coprod_{X'_0} X'_2$$

where:

- The space X'_0 is the disjoint union of the spaces $\text{sk}^i X$.
- The space X'_1 is the disjoint union of the mapping cylinders for the inclusions $\text{sk}^i X \subseteq \text{sk}^{i+1} X$, where i is odd.
- The space X'_2 is the disjoint union of the mapping cylinders for the inclusions $\text{sk}^i X \subseteq \text{sk}^{i+1} X$, where i is even.

Since X is atomic, the equivalence $X \simeq X'$ factors (up to homotopy) through either X'_1 or X'_2 . Since X is connected, we this map factors through the mapping cylinder of the inclusion $\text{sk}^i X \subseteq \text{sk}^{i+1} X$, for some integer i . Consequently, we deduce that the identity map from X to itself factors up to homotopy through the finite dimensional spaces $\text{sk}^{i+1} X$.

We now prove, by induction descending induction on j , that the identity map id_X factors (up to homotopy) through $\text{sk}^j X$. The case $j = i + 1$ follows from the above argument. For the inductive step, we use the homotopy pushout diagram

$$\begin{array}{ccc} \coprod S^{j-1} & \longrightarrow & \text{sk}^{j-1} X \\ \downarrow & & \downarrow \\ \coprod D^j & \longrightarrow & \text{sk}^j X. \end{array}$$

Since X is atomic, we conclude that id_X factors (up to homotopy) either through $\text{sk}^{j-1} X$ or through $\coprod D^j$. Since X is connected, the latter possibility implies that id_X factors up to homotopy through some disk D^j , which contradicts our assumption that X is not contractible. Therefore id_X factors through $\text{sk}^{j-1} X$.

Applying the above argument repeatedly, we deduce that id_X factors through the $\text{sk}^{-1} X = \emptyset$. Since X is nonempty, we obtain a contradiction. \square

Where does the above argument go wrong if we work in the p -profinite category?