18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Sullivan Conjecture Revisited (Lecture 33)

In this lecture we will prove the following version of the Sullivan conjecture:

Theorem 1. *Let X be a simply connected finite cell complex, and let G be a finite group. Then the diagonal inclusion*

$$X \to X^{BG}$$

is a weak homotopy equivalence.

In the last lecture, we saw that X fits into a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & \prod \widehat{X}_p \\ \downarrow & & \downarrow \\ X_{\mathbf{Q}} & \longrightarrow & (\prod \widehat{X}_p)_{\mathbf{Q}}. \end{array}$$

Let us say that a space Y is *good* if, for every finite group G, the diagonal map $Y \to Y^{BG}$ is a weak homotopy equivalence. The collection of good spaces is obviously stable under homotopy limits. Consequently, Theorem 1 is an immediate consequence of the following:

Proposition 2. *Let X be a simply connected finite cell complex. Then:*

(1) *For every prime p, the p-adic completion $\widehat{X}_p$ is good.*

(2) *The rationalization $X_{\mathbf{Q}}$ is good.*

(3) *The "adelic completion" $(\prod_p \widehat{X}_p)_{\mathbf{Q}}$ is good.*

Assertions (2) and (3) follow from the following more general statement:

Lemma 3. *Let Y be a rational space. Then Y is good.*

Proof. We wish to show that the map

$$\mathrm{Map}(*, Y) \to \mathrm{Map}(BG, Y)$$

is a homotopy equivalence, for every finite group G. Since Y is rational, it will suffice to show that the projection $BG \to *$ is a rational homotopy equivalence. In other words, we must show that $\mathrm{H}^*(BG; \mathbf{Q})$ vanishes for $* > 0$. This is clear: the higher cohomology of a finite group G is annihilated by the order $|G|$ of G. $\square$

We now focus on the proof of part (1) in Proposition 2. Fix a prime number p. We will begin by studying the situation where the finite group G is a p-group. In this case, we have

$$\begin{aligned} (\widehat{X}_p)^{BG} &\simeq (\varprojlim X_p^{\vee})^{BG} \\ &\simeq \varprojlim ((X_p^{\vee})^{BG}). \end{aligned}$$

Since X is finite dimensional, our p-profinite version of the Sullivan conjecture implies that the canonical map $X_p^\vee \to (X_p^\vee)^{BG}$ is an equivalence of p-profinite spaces. Passing to the homotopy inverse limit, we get a homotopy equivalence

$$\widehat{X}_p \to (\widehat{X}_p)^{BG},$$

as desired.

Now let G be an arbitrary finite group. Let H be a p-Sylow subgroup of G. We have a canonical map $BH \to BG$; without loss of generality, we may arrange that this is a covering map whose fibers can be identified with the finite set G/H. We define a simplicial space $K_\bullet$ by the formula

$$K_n = BH \times_{BG} BH \times_{BG} \times \ldots \times_{BG} BH,$$

where the factor BH appears $(n+1)$-times. We have a canonical homotopy equivalence

$$|K_\bullet| \to BG.$$

We can describe the space $K_\bullet$ more carefully as follows. Let $M_\bullet$ be the simplicial set with $M_n = (G/H)^{n+1}$. Then G acts (diagonally) on the simplicial set $M_\bullet$, and the simplicial space $K_\bullet$ can be identified with the homotopy quotient $(M_\bullet)_{hG}$. Let $K'_\bullet$ be the simplicial set defined by the formula

$$K'_n = \pi_0 K_n,$$

so that $K'_\bullet$ can be identified with the ordinary quotient $(M_\bullet)_G$. We can identify an element of K'_n with an equivalence class of sequences $(g_0H, \ldots, g_nH)$, where each c_i is a (right) coset of H in G, and two sequences $(g_0H, \ldots, g_nH)$ and $(g'_0H, \ldots, g'_nH)$ are equivalence if there exists an element $g \in G$ such that $g_iH = gg'_iH$ for $0 \leq i \leq n$.

For each n, the fiber of the map $K_n \to K'_n$ over an n-tuple $(g_0H, \ldots, g_nH)$ can be identified with the classifying space BP, where $P = g_0Hg_0^{-1} \cap g_1Hg_1^{-1} \cap \ldots \cap g_nHg_n^{-1}$. In particular, P is conjugate to a subgroup of H, and is therefore a finite p-group. It follows that the diagonal map $\widehat{X}_p \to (\widehat{X}_p)^{BP}$ is a homotopy equivalence. Taking a product over all elemenets of K'_n, we conclude that the map

$$(\widehat{X}_p)^{K'_n} \to (\widehat{X}_p)^{K_n}$$

is a homotopy equivalence.

We now compute

$$\begin{aligned} (\widehat{X}_p)^{BG} &\simeq (\widehat{X}_p)^{|K_\bullet|} \\ &\simeq \varprojlim (\widehat{X}_p)^{K_n} \\ &\simeq \varprojlim (\widehat{X}_p)^{K'_n} \\ &\simeq (\widehat{X}_p)^{|K'_\bullet|}. \end{aligned}$$

It will therefore suffice to show that the diagonal map

$$\widehat{X}_p \to \mathrm{Map}(|K'_\bullet|, \widehat{X}_p)$$

is a homotopy equivalence. Since $\widehat{X}_p$ is an $\mathbf{F}_p$-local space, this is an immediate consequence of the following lemma:

Lemma 4. *The projection $|K'_\bullet| \to *$ induces an equivalence on $\mathbf{F}_p$-homology.*

In other words, we claim that the homology groups $\mathrm{H}_*(|K'_\bullet|; \mathbf{F}_p)$ vanish for $* > 0$. These are the homology groups of the complex

$$\ldots \to \mathbf{F}_p[K'_2] \to \mathbf{F}_p[K'_1] \to \mathbf{F}_p[K'_0] \to 0,$$

where $\mathbf{F}_p[Z]$ denotes the free $\mathbf{F}_p$-vector space on a basis given by the elements of $\mathbf{Z}$. The simplicial set $K'_\bullet$ can be extended to an *augmented* simplicial set by defining $K'_{-1} = * \simeq ((G/H)^0)_G$, so we get an augmented chain complex

$$\ldots \to \mathbf{F}_p[K'_2] \to \mathbf{F}_p[K'_1] \to \mathbf{F}_p[K'_0] \to \mathbf{F}_p[K'_{-1}] \to 0.$$

We will show that this chain complex is acyclic (in all degrees). For this, it suffices to exhibit a contracting chain homotopy h. We choose a homotopy h given by the formula

$$(g_0H, \ldots, g_nH) \mapsto \frac{1}{|G/H|} \sum_{g \in G/H} (gH, g_0H, \ldots, g_nH).$$

This map is well-defined since it is clearly G-invariant, and the expression $\frac{1}{|G/H|}$ makes sense in virtue of our assumption that H is a p-Sylow subgroup of G. A simple calculation shows that this map is indeed a contracting homotopy. This completes the proof of Theorem 1.

Remark 5. We have assumed that X is a simply connected finite CW complex. This assumption was used in two ways:

(1) We invoked the fact that X was simply connected and that the homotopy groups $\pi_i X$ are finitely generated, in order to use the arithmetic square discussed in the previous lecture.

(2) We invoked the fact that X was finite dimensional so that we could appeal to our p-profinite version of the Sullivan conjecture.

Assumptions (1) and (2) guarantee that X is a finite complex, at least up to homotopy equivalence. But Haynes Miller's original proof of Theorem 1 actually works in a much more general setting: one only needs to assume that X is finite dimensional (in particular, the fundamental group $\pi_1 X$ can be arbitrary).