18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Quaternionic Projective Space (Lecture 34)

The three-sphere S^3 can be identified with $SU(2)$, and therefore has the structure of a topological group. In this lecture, we will address the question of how canonical this structure is. In the category of topological groups, the group structure on S^3 is unique up to isomorphism. However, the purely homotopy-theoretic situation is not quite so nice: there exist uncountably many pairwise inequivalent group structures on spaces which are homotopy equivalent to S^3 (we will return to this point at the end of the lecture). However, the situation is much simpler in p-adic homotopy theory, where p is a fixed prime. In this case, we again have a unique group structure on (the p-adic completion) of the homotopy type of S^3. We will sketch a proof of this result when p is odd, following the ideas of Dwyer, Miller and Wilkerson.

We begin by formulating the problem more precisely. In homotopy theory, giving a group structure on a homotopy type G is equivalent to realizing G as the loop space of a pointed space X. In this case, we have a fiber sequence

$$G \to * \to X.$$

If $G = S^3$, then we can use the Serre spectral sequence to compute the (mod p) cohomology ring of X: $\mathrm{H}^*(X) \simeq \mathbf{F}_p[t]$, where t lies in degree 4 (and transgresses to the fundamental class of $G = S^3$). Moreover, we have the same picture in the p-profinite category. We can now state the main result:

Theorem 1 (Dwyer, Miller, Wilkerson)**.** *Let X be a p-profinite space such that $\mathrm{H}^*(X) \simeq \mathbf{F}_p[t]$, where t lies in degree 4. Then X is equivalent to the p-profinite completion $BSU(2)_p^\vee$ of the classifying space of the group $SU(2)$ (in other words, infinite dimensional quaternionic projective space).*

The first step is to describe the cohomology $\mathrm{H}^*(X)$ as a representation of the mod-p Steenrod algebra $\mathcal{A}_p$. To simplify the exposition, we will consider only the case $p \neq 2$. We therefore begin with a few recollections on the structure of $\mathcal{A}_p$:

- For any space X (or any p-profinite space), the algebra $\mathcal{A}_p$ acts on the cohomology ring $\mathrm{H}^*(X; \mathbf{F}_p)$.
- The algebra $\mathcal{A}_p$ is generated the Bockstein operator β of degree 1, together with operations P^i of degree $2i(p-1)$, for $i > 0$.
- We have a Cartan formula
$$P^n(xy) = \sum_{n=n'+n''} P^{n'}(x)P^{n''}(y),$$
and a similar formula for β (which involves a sign). Here we agree by convention that $P^0 = \mathrm{id}$.
- If $x \in \mathrm{H}^{2i}(X; \mathbf{F}_p)$, then $P^i(x) = x^p$ and $P^j(x) = 0$ for $j > i$ (instability).
- We have $P^1P^1 = 2P^2$ (this is a special case of the Adem relations, which we will not write out in full).

Lemma 2. *Let X be as in the statement of Theorem 1. Then there exists an isomorphism $\alpha : \mathrm{H}^*(X) \simeq \mathbf{F}_p[t]$ such that the action of $\mathcal{A}_p$ on $\mathrm{H}^*(X) \simeq \mathbf{F}_p[t]$ is determined by the Cartan formula, together with the relations*

$$\beta t = 0$$

$$P^i t = \begin{cases} 2t^{\frac{p+1}{2}} & \textit{if } i = 1 \\ t^p & \textit{if } i = 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The formula $\beta t = 0$ is obvious, since $\mathrm{H}^5(X) \simeq 0$. The expressions $P^i t$ vanishes for $i > 2$ by instability, and $P^2 t = t^p$. We have $P^1(t) = ct^{\frac{p+1}{2}}$ for some constant $c \in \mathbf{F}_p$; the only nontrivial point is to compute c. For this, we observe

$$\begin{aligned} 2t^p &= 2P^2(t) \\ &= P^1 P^1(t) \\ &= cP^1 t^{\frac{p+1}{2}} \\ &= c^2 \frac{p+1}{2} t^p \end{aligned}$$

so that $c^2 = \frac{4}{p+1} = 4$. This has solutions $c = \pm 2$. However, if $c = -2$ then we can adjust the isomorphism α via the substitution $t \mapsto \lambda t$, where $\lambda \in \mathbf{F}_p$ is not a quadratic residue, to obtain an isomorphism with the desired property. □

Corollary 3. *There exists an isomorphism* $\alpha : \mathrm{H}^*(X) \simeq \mathrm{H}^*(BSU(2))$ *of unstable* $\mathcal{A}_p$*-algebras.*

We now make a few remarks about the structure of the group $SU(2)$. We have injective group homomorphisms

$$\mathbf{Z}/p\mathbf{Z} \hookrightarrow S^1 \hookrightarrow SU(2).$$

These induce maps of classifying spaces

$$B\mathbf{Z}/p\mathbf{Z} \to BS^1 \to BSU(2),$$

hence we get maps on cohomology

$$\mathrm{H}^*(B\mathbf{Z}/p\mathbf{Z}) \leftarrow \mathrm{H}^*(BS^1) \leftarrow \mathrm{H}^*(BSU(2)).$$

A simple computation shows that each of these maps is injective, and we can identify the above with the sequence

$$\mathbf{F}_p[u, \epsilon] \hookleftarrow \mathbf{F}_p[u] \hookleftarrow \mathbf{F}_p[t].$$

Here $t \mapsto u^2$, where u has degree 2, and ϵ has degree 1 in $\mathrm{H}^*(B\mathbf{Z}/p\mathbf{Z})$ (and therefore squares to zero).

Lemma 4. *There exists a map* $\beta : B\mathbf{Z}/p\mathbf{Z} \to X$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{H}^*(X) & & \\ \downarrow{\scriptstyle \alpha} & \searrow & \\ & & \mathrm{H}^*(B\mathbf{Z}/p\mathbf{Z}) \\ & \nearrow & \\ \mathrm{H}^*(BSU(2)) & & \end{array}$$

commutes.

Proof. We have

$$\begin{aligned}
\pi_0 \operatorname{Map}(B\mathbf{Z}/p\mathbf{Z}, X) &\simeq \operatorname{Hom}(\mathrm{H}^*(X^{B\mathbf{Z}/p\mathbf{Z}}), \mathbf{F}_p) \\
&\simeq \operatorname{Hom}(T\,\mathrm{H}^*(X), \mathbf{F}_p) \\
&\simeq \operatorname{Hom}(\mathrm{H}^*(X), \mathrm{H}^*(B\mathbf{Z}/p\mathbf{Z}))
\end{aligned}$$

Here the Hom-sets on the right hand side are computed in the category of unstable $\mathcal{A}_p$-algebras. In other words, any map of $\mathcal{A}_p$-algebras from $\mathrm{H}^*(X)$ to $\mathrm{H}^*(B\mathbf{Z}/p\mathbf{Z})$ is necessarily induced by a map of p-profinite spaces $B\mathbf{Z}/p\mathbf{Z}$ to X (which is then uniquely determined up to homotopy). □

Let Y be the connected component of the mapping space $X^{B\mathbf{Z}/p\mathbf{Z}}$ containing the map β. We then have isomorphisms

$$\begin{aligned}
\mathrm{H}^*(Y) &\simeq \mathrm{H}^*(X^{B\mathbf{Z}/p\mathbf{Z}}) \otimes_{\mathrm{H}^0(X^{B\mathbf{Z}/p\mathbf{Z}})} \mathbf{F}_p \\
&\simeq T\,\mathrm{H}^*(X) \otimes_{(T\,\mathrm{H}^*(X))^0} \mathbf{F}_p.
\end{aligned}$$

Consequently, the cohomology ring $\mathrm{H}^*(Y)$ depends only on $\mathrm{H}^*(X)$.

Let us temporarily assume that $X = BSU(2)^\vee_p$ and that β is the map induced by the group homomorphism $\mathbf{Z}/p\mathbf{Z} \to SU(2)$. The loop space ΩY can be identified with the space of homotopies from β to itself, which is a space of sections of a certain fibration

$$E \to B\mathbf{Z}/p\mathbf{Z}$$

with fiver $SU(2)^\vee_p$. This fibration corresponds to an action of $\mathbf{Z}/p\mathbf{Z}$ on $SU(2)^\vee_p$, which is simply induced by the action of $\mathbf{Z}/p\mathbf{Z}$ by conjugation. We therefore may therefore identify ΩY with the homotopy fixed set $(SU(2)^\vee_p)^{h\mathbf{Z}/p\mathbf{Z}}$. Using the p-profinite Sullivan conjecture, this can be identified with the p-profinite completion of the actual fixed set $SU(2)^{\mathbf{Z}/p\mathbf{Z}}$, which is simply the centralizer of $\mathbf{Z}/p\mathbf{Z}$ in $SU(2)$. A simple calculation shows that this centralizer coincides with the circle group $S^1 \subseteq SU(2)$. It follows that $\Omega Y \simeq (S^1)^\vee_p$. Using the Serre spectral sequence, we conclude that $\mathrm{H}^*(Y)$ is isomorphic to $\mathbf{F}_p[u]$, where u lies in degree 2. Moreover, the translation action of $B\mathbf{Z}/p\mathbf{Z}$ on itself determines a map $B\mathbf{Z}/p\mathbf{Z} \to Y$, which (after scaling u if necessary) is given on cohomology by the canonical inclusion

$$\mathbf{F}_p[u] \hookrightarrow \mathbf{F}_p[u, \epsilon].$$

We now return to the general case. Since $\mathrm{H}^*(Y)$ depends only on $\mathrm{H}^*(X)$, we conclude that $\mathrm{H}^* \simeq \mathbf{F}_p[u]$ in general. Evaluation at the base point of $B\mathbf{Z}/p\mathbf{Z}$ induces a map $e : Y \to X$. Moreover, the composition

$$B\mathbf{Z}/p\mathbf{Z} \to Y \xrightarrow{e} X$$

can be identified with the map β. It follows that the above sequence induces, on cohomology, the maps

$$\mathbf{F}_p[u, \epsilon] \hookleftarrow \mathbf{F}_p[u] \hookleftarrow \mathbf{F}_p[t].$$

Consider the map from $X^{B\mathbf{Z}/p\mathbf{Z}}$ to itself, given by composition with the map

$$\mathbf{Z}/p\mathbf{Z} \xrightarrow{-1} \mathbf{Z}/p\mathbf{Z}.$$

This map induces the identify on $\mathrm{H}^4(B\mathbf{Z}/p\mathbf{Z})$, and therefore induces the identity map on $\operatorname{Hom}(\mathrm{H}^*(X), \mathrm{H}^*(B\mathbf{Z}/p\mathbf{Z})) \simeq \pi_0 X^{B\mathbf{Z}/p\mathbf{Z}}$. It therefore induces an involution on Y, which we will denote by i. We have a commutative diagram

$$\begin{array}{ccc}
B\mathbf{Z}/p\mathbf{Z} & \longrightarrow & Y \\
\downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle i} \\
B\mathbf{Z}/p\mathbf{Z} & \longrightarrow & Y,
\end{array}$$

which gives a commutative diagram of cohomology groups

$$\begin{array}{ccc} \mathbf{F}_p[u,\epsilon] & \longleftarrow & \mathbf{F}_p[u] \\ \uparrow & & \uparrow \\ \mathbf{F}_p[u,\epsilon] & \longleftarrow & \mathbf{F}_p[u] \end{array}$$

Since the left vertical map carries u to $-u$, the right vertical map does as well. Let $Y_{h\mathbf{Z}/2\mathbf{Z}}$ denote the homotopy coinvariants of the involution on Y. Then the canonical map $Y \to Y_{h\mathbf{Z}/2\mathbf{Z}}$ induces an isomorphism

$$\mathrm{H}^*(Y_{h\mathbf{Z}/2\mathbf{Z}}) \simeq \mathrm{H}^*(Y)^{\mathbf{Z}/2\mathbf{Z}} \simeq \mathbf{F}_p[u^2].$$

The base point of $B\mathbf{Z}/p\mathbf{Z}$ is invariant under the map given by multiplication by (-1), so the evaluation map $e : Y \to X$ is invariant under the action of i. Consequently, we obtain a factorization

$$\begin{array}{ccccc} Y & & \xrightarrow{\quad e \quad} & & X \\ & \searrow & & \nearrow_{e'} & \\ & & Y_{h\mathbf{Z}/2\mathbf{Z}}. & & \end{array}$$

This induces a commutative diagram of cohomology groups

$$\begin{array}{ccccc} \mathbf{F}_p[u] & & \longleftarrow & & \mathbf{F}_p[t] \\ & \nwarrow & & \swarrow & \\ & & \mathbf{F}_p[u^2]. & & \end{array}$$

We conclude that e' induces an isomorphism on cohomology, and therefore a homotopy equivalence of p-profinite spaces $Y_{h\mathbf{Z}/2\mathbf{Z}} \to X$.

We now identify the p-profinite space Y. Since the cohomology of Y lies entirely in even degrees, we can choose a compatible family of cohomology classes $u_i \in \mathrm{H}^2(Y; \mathbf{Z}/p^i\mathbf{Z})$ lifting u. These cohomology classes determine a map of p-profinite spaces

$$Y \to \text{“}\varprojlim K(\mathbf{Z}/p^k, 2)\text{”},$$

which we can identify with a map $Y \to (BS^1)^\vee_p$. This map induces an isomorphism on cohomology, and is therefore an equivalence of p-profinite spaces. We may therefore identify Y with the (p-profinite) Eilenberg-MacLane space $K(\mathbf{Z}_p, 2)$.

Now consider the involution i on Y. We claim that the homotopy fixed set $Y^{h\mathbf{Z}/2\mathbf{Z}}$ is nonempty: this follows from the vanishing of the cohomology group $\mathrm{H}^3(B\mathbf{Z}/2\mathbf{Z}; \mathbf{Z}_p)$ (since p is different from 2). We may therefore assume without loss of generality that Y contains a point fixed by the involution i. In this case, i can be regarded as a pointed map from the Eilenberg-MacLane space $K(\mathbf{Z}_p, 2)$ to itself, which is given by a group homomorphism $h : \mathbf{Z}_p \to \mathbf{Z}_p$. Since h has order 2, we deduce that h is given by the formula $h(z) = \lambda z$, where $\lambda = \pm 1$. Since i carries $u \in \mathrm{H}^2(Y)$ to $-u$, we deduce that $\lambda = -1$. We have therefore proven:

Theorem 5. *Let X be as in Theorem 1 and p an odd prime. Then there is an equivalence of p-profinite spaces*

$$X \simeq K(\mathbf{Z}_p, 2)_{h\mathbf{Z}/2\mathbf{Z}},$$

where the group $\mathbf{Z}/2\mathbf{Z}$ acts on $\mathbf{Z}_p$ by the sign involution.

In particular, there is only one possibility for the homotopy type of X. Theorem 1 follows.

Let us now consider the same problem in the non-p-profinite world. Let X be a simply connected space such that $\mathrm{H}^*(X; \mathbf{Z}) \simeq \mathrm{H}^*(BSU(2); \mathbf{Z}) \simeq \mathbf{Z}[t]$, where t lies in degree 4 (this is equivalent to the assertion that the loop space ΩX is homotopy equivalent to a three sphere S^3, by the Serre spectral sequence). We have a homotopy pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & \prod_p \widehat{X}_p \\ \big\downarrow & & \big\downarrow \\ X_{\mathbf{Q}} & \longrightarrow & (\prod_p \widehat{X}_p)_{\mathbf{Q}}. \end{array}$$

Using Theorem 1 (and its analogue in the case $p = 2$), we deduce that for each prime p we have a homotopy equivalence $\widehat{X}_p \simeq \widehat{BSU(2)}_p$. A much easier argument shows that $X_{\mathbf{Q}} \simeq K(\mathbf{Q}, 4) \simeq BSU(2)_{\mathbf{Q}}$. We can therefore rewrite the above homotopy pullback diagram as

$$\begin{array}{ccc} X & \longrightarrow & \prod_p \widehat{BSU(2)}_p \\ \big\downarrow & & \big\downarrow \\ BSU(2)_{\mathbf{Q}} & \xrightarrow{\phi} & (\prod_p \widehat{BSU(2)}_p)_{\mathbf{Q}}. \end{array}$$

However, this does not imply that $X \simeq BSU(2)$, because the map ϕ has not been determined. The domain of ϕ can be identified with an Eilenberg-MacLane space $K(\mathbf{Q}, 4)$, and the codomain of ϕ with an Eilenberg-MacLane space $K((\prod_p \mathbf{Z}_p)_{\mathbf{Q}}, 4)$, so that ϕ is determined up to homotopy by specifying an element $\eta \in (\prod_p \mathbf{Z}_p)_{\mathbf{Q}}$. Every invertible element $\eta \in (\prod_p \mathbf{Z}_p)_{\mathbf{Q}}$ gives rise to a space X which is a delooping of the sphere S^3. Not all of these choices are distinct (as an exercise, you can try to figure out when two choices of η give homotopy equivalent deloopings), but this "mixing" construction nevertheless yields uncountably many group structures on the homotopy type S^3.