18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Nil-Filtration (Lecture 36)

In the last lecture, we showed that the category $\mathcal{U}$ of unstable Steenrod modules fits into an adjunction

$$\mathcal{U} \underset{g_n}{\overset{f_n}{\rightleftarrows}} \mathrm{Fun}_n,$$

where f_n is exact and g_n is fully faithful. Our goal in this lecture is to put this result into a more general context.

Definition 1. Let $\mathcal{C}$ be a Grothendieck abelian category. A *Serre class* in $\mathcal{C}$ is a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ such that:

(1) Given a short exact sequence
$$0 \to X' \to X \to X'' \to 0$$
in $\mathcal{C}$, the object X belongs to $\mathcal{C}_0$ if and only if X' and X'' belong to $\mathcal{C}_0$.

(2) The subcategory $\mathcal{C}_0$ is closed under small colimits in $\mathcal{C}$ (in virtue of (1), this is equivalent to being closed under direct sums).

(3) The abelian category $\mathcal{C}_0$ is Grothendieck: in other words, there exists a set of objects of $\mathcal{C}_0$ which generates $\mathcal{C}_0$ under colimits.

We say that a morphism $f : X \to Y$ in $\mathcal{C}$ is a $\mathcal{C}_0$-*equivalence* if the kernel and cokernel of f belong to $\mathcal{C}_0$.

In what follows, we fix a Grothendieck abelian category $\mathcal{C}$ and a Serre subcategory $\mathcal{C}_0$.

Lemma 2. *Let X be an object of $\mathcal{C}$. The following conditions are equivalent:*

(1) *For every $\mathcal{C}_0$-equivalence $Y \to Y'$, the induced map $\mathrm{Hom}_{\mathcal{C}}(Y', X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a bijection.*

(2) *For every object $Z \in \mathcal{C}_0$, we have $\mathrm{Hom}_{\mathcal{C}}(Z, X) = \mathrm{Ext}_{\mathcal{C}}(Z, X) = 0$.*

Proof. Suppose first that (1) is satisfied. If $Z \in \mathcal{C}_0$, then the map $0 \to Z$ is a $\mathcal{C}_0$-equivalence, so we get $\mathrm{Hom}_{\mathcal{C}}(Z, X) \simeq \mathrm{Hom}_{\mathcal{C}}(0, X) \simeq 0$. To prove that $\mathrm{Ext}_{\mathcal{C}}(Z, X)$ vanishes, we consider an arbitrary extension

$$0 \to X \xrightarrow{f} Y \to Z \to 0$$

and show that it is split. The map f is a $\mathcal{C}_0$-equivalence, so composition with f induces a bijection $\mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(X, X)$. In particular, the identity map from X to itself factors through f, so the above exact sequence splits.

Now suppose that (2) is satisfied, and let $g : Y \to Y'$ be a $\mathcal{C}_0$-equivalence. Then g factors as a composition

$$Y \xrightarrow{g'} \mathrm{Im}(g) \xrightarrow{g''} Y',$$

where g' is an epimorphism and g'' is a monomorphism. We may therefore assume that g is either epic or monic. In the epic case, we have a short exact sequence

$$0 \to \ker(g) \to Y \to Y' \to 0$$

which yields an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(Y', Z) \to \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(\ker(g), Z) = 0.$$

In the monic case, we have a short exact sequence

$$0 \to Y \to Y' \to \mathrm{coker}(g) \to 0$$

which gives rise to an exact sequence

$$0 \simeq \mathrm{Hom}_{\mathcal{C}}(\mathrm{coker}(g), Z) \to \mathrm{Hom}_{\mathcal{C}}(Y', Z) \to \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Ext}_{\mathcal{C}}(\mathrm{coker}(g), Z) \simeq 0.$$

□

Definition 3. We will say that an object $X \in \mathcal{C}$ is $\mathcal{C}_0$*-local* if the equivalent conditions of Lemma 2 are satisfied. We let $\mathcal{C} / \mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ consisting of $\mathcal{C}_0$-local objects.

Example 4. Let $\mathcal{C}$ be the category of abelian groups, and $\mathcal{C}_0$ the full subcategory consisting of abelian groups M such that every element $m \in M$ satisfies $p^k m = 0$ for $k \gg 0$. Then $\mathcal{C}_0$ is a Serre class in $\mathcal{C}$. An abelian group is $\mathcal{C}_0$-local if and only if it is a module over the ring $\mathbf{Z}[\frac{1}{p}]$.

Example 5. Let $\mathcal{U}$ be the category of unstable modules over the Steenrod algebra $\mathcal{A}$, and let $\mathrm{Nil} \subseteq \mathcal{U}$ denote the subcategory of *nilpotent* modules. Then Nil is a Serre class in $\mathcal{U}$.

Remark 6. It is clear from characterization (1) of Lemma 2 that the collection of $\mathcal{C}_0$-local objects of $\mathcal{C}$ is stable under arbitrary limits.

Proposition 7. *Let $\mathcal{C}$ be a Grothendieck abelian category and $\mathcal{C}_0 \subseteq \mathcal{C}$ a Serre class. Then:*

(1) *The inclusion $\mathcal{C} / \mathcal{C}_0 \subseteq \mathcal{C}$ admits a left adjoint L.*

(2) *The category $\mathcal{C} / \mathcal{C}_0$ is a Grothendieck abelian category.*

(3) *The functor L is exact.*

Warning 8. The inclusion $\mathcal{C} / \mathcal{C}_0 \subseteq \mathcal{C}$ is *not* an exact functor in general. The formation of cokernels in $\mathcal{C} / \mathcal{C}_0$ is given by first forming cokernels in $\mathcal{C}$, and then applying the functor L.

Proof. Using the small object argument, one can show that every object $X \in \mathcal{C}$ admits a $\mathcal{C}_0$-equivalence $X \to LX$, where LX is $\mathcal{C}_0$-local. One can then show that LX depends functorially on X and yields the desired adjoint.

We will prove (2). First, we show that $\mathcal{A} = \mathcal{C} / \mathcal{C}_0$ is an abelian category. It is easy to see that $\mathcal{A}$ is additive and admits kernels and cokernels. To avoid confusion, if $f : X \to Y$ is a morphism in $\mathcal{A}$, we let $\mathrm{coker}_{\mathcal{A}}(f)$ denote the cokernel of f in the category $\mathcal{A}$, and $\mathrm{coker}_{\mathcal{C}}(f)$ its cokernel in the category $\mathcal{C}$, so that we have an identification $\mathrm{coker}_{\mathcal{A}}(f) \simeq L\,\mathrm{coker}_{\mathcal{C}}(f)$. (There is no need to introduce any complicated notation for kernels, since these can be computed either in $\mathcal{A}$ or in $\mathcal{C}$.) To prove that $\mathcal{A}$ is an abelian category, we must show that if $f : X \to Y$ is a morphism in $\mathcal{A}$, then the canonical map

$$\mathrm{coker}_{\mathcal{A}}(\ker(f) \to X) \to \ker(Y \to \mathrm{coker}_{\mathcal{A}}(f))$$

is an isomorphism. In other words, we must show that the map

$$L\,\mathrm{coker}_{\mathcal{C}}(\ker(f) \to X) \to \ker(Y \to L\,\mathrm{coker}_{\mathcal{C}}(f))$$

is an equivalence. This is equivalent to showing that the map

$$\phi : \mathrm{coker}_{\mathcal{C}}(\ker(f) \to X) \to \ker(Y \to L\,\mathrm{coker}_{\mathcal{C}}(f))$$

is a $\mathcal{C}_0$-equivalence. Since $\mathcal{C}$ is an abelian category, we can identify the left hand side with $\ker(Y \to \mathrm{coker}_{\mathcal{C}}(f))$. We have a short exact sequence

$$0 \to \ker(Y \to \mathrm{coker}_{\mathcal{C}}(f)) \xrightarrow{\phi} \ker(Y \to L\,\mathrm{coker}_{\mathcal{C}}(f)) \to \ker(\mathrm{coker}_{\mathcal{C}}(f) \to L\,\mathrm{coker}_{\mathcal{C}}(f).$$

The desired result now follows, since $\mathrm{coker}(\phi)$ is a subobject of an object of $\mathcal{C}_0$, and therefore belongs to $\mathcal{C}_0$.

Assuming (3) for the moment, we now show that $\mathcal{A}$ is a Grothendieck abelian category. The existence of colimits and a set of generators follows from general categorical nonsense. It therefore suffices to show that filtered colimits are exact. In other words, we must show that if $\{f_\alpha X_\alpha \to Y_\alpha\}$ is a filtered diagram of monomorphisms in $\mathcal{A}$, then the colimit $\varinjlim_{\mathcal{A}}\{f_\alpha\}$ is a monomorphism. We have

$$\varinjlim_{\mathcal{A}}\{f_\alpha\} \simeq L \varinjlim_{\mathcal{C}}\{f_\alpha\},$$

so the desired result follows from the exactness of L and the assumption that $\mathcal{C}$ is Grothendieck.

We now prove (3). Since L is a left adjoint, it is automatically right exact. It will therefore suffice to prove that L preserves monomorphisms. Let $f : X \to Y$ be a monomorphism in $\mathcal{C}$; we wish to prove that $Lf : LX \to LY$ is again a monomorphism. Let $K = \ker(Lf)$, and let $K' = K \times_{LX} X \subseteq X$. Since f is a monomorphism, f induces a monomorphism

$$K' \to \ker(\alpha) \subseteq Y,$$

where $\alpha : Y \to LY$ is the canonical map. Since α is a $\mathcal{C}_0$-equivalence, we deduce that $K' \in \mathcal{C}_0$. We have an exact sequence

$$K' \to K \to \mathrm{coker}(X \to LX),$$

so that $K \in \mathcal{C}_0$ as well. But then the inclusion $K \subseteq LX$ must be the zero map, so that $K \simeq 0$ as desired. □

The next result shows that $\mathcal{C}/\mathcal{C}_0$ can really be viewed as a "quotient" of $\mathcal{C}$ by $\mathcal{C}_0$:

Proposition 9. *Let $\mathcal{D}$ be a Grothendieck abelian category, and $F : \mathcal{C} \to \mathcal{D}$ a colimit-preserving functor. Then:*

(1) *The functor F is isomorphic to a composition*

$$\mathcal{C} \xrightarrow{L} \mathcal{C}/\mathcal{C}_0 \xrightarrow{F'} \mathcal{D}$$

if and only if F carries $\mathcal{C}_0$-equivalences to isomorphisms in $\mathcal{D}$. Moreover, in this case, F' is determined up to unique isomorphism (and is colimit preserving).

(2) *The functor F' is exact if and only if F is exact.*

Proof. Note that $F' = F|\,\mathcal{C}/\mathcal{C}_0$ is, up to isomorphism, the only functor satisfying the condition of (1); the condition that $F \simeq F \circ L$ is equivalent to the requirement that F carries $\mathcal{C}_0$-equivalences to isomorphisms. This proves (1). We now prove (2). The "only if" direction is clear, since L is exact. Conversely, suppose that F is exact. Since F' preserves colimits, it is automatically right exact; it therefore suffices to show that F' preserves monomorphisms. This follows from the exactness of F, since $F' \simeq F|\,\mathcal{C}/\mathcal{C}_0$ and a morphism $f : X' \to X$ is a monomorphism in $\mathcal{C}$ if and only if it is a monomorphism in $\mathcal{C}/\mathcal{C}_0$. This proves (2). □

Remark 10. Note that, if F is exact, then F carries $\mathcal{C}_0$-equivalences to isomorphisms if and only if F annihilates every object of $\mathcal{C}_0$.

Corollary 11. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact, colimit preserving functor between Grothendieck abelian categories. Then:*

(1) *Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory consisting of objects $X \in \mathcal{C}$ such that $FX \simeq 0$. Then $\mathcal{C}_0$ is a Serre class in $\mathcal{C}$.*

(2) *The functor F factors as a composition $\mathcal{C} \to \mathcal{C} / \mathcal{C}_0 \xrightarrow{F'} \mathcal{D}$, where F' is an exact colimit preserving functor.*

(3) *The functor F admits a right adjoint G.*

(4) *The functor F' is an equivalence if and only if G is fully faithful.*

Proof. Assertion (1) follows immediately from the definitions, and (2) follows from Proposition 9. Assertion (3) follows from the adjoint functor theorem. The "only if" direction of (4) is clear, since the localization functor $L : \mathcal{C} \to \mathcal{C} / \mathcal{C}_0$ is left adjoint to the fully faithful inclusion $\mathcal{C} / \mathcal{C}_0 \subseteq \mathcal{C}$. For the converse, let us suppose that G is fully faithful. Replacing $\mathcal{C}$ by $\mathcal{C} / \mathcal{C}_0$ if necessary, we may reduce to the case $\mathcal{C}_0 = 0$. We wish to show that F is an equivalence of categories. Since G is fully faithful, the counit map $\beta_D : FG(D) \to D$ is an isomorphism for any $D \in \mathcal{D}$. We want to show that the unit map $\alpha : C \to GF(C)$ is an isomorphism for each $C \in \mathcal{C}$. The map $F(\alpha)$ is a right inverse to the invertible morphism $\beta_{FC} : FG(F(C)) \to F(C)$, so $F(\alpha)$ is an isomorphism. It follows that $\ker(\alpha)$ and $\operatorname{coker}(\alpha)$ are annihilated by F, so $\ker(\alpha) \simeq \operatorname{coker}(\alpha) \simeq 0$ and α is an isomorphism as desired. □

We now return to our main example:

Corollary 12. *Let $f_n : \mathcal{U} \to \mathrm{Fun}_n$ be the functor defined in the last lecture, so that $f_n(M)(V) = \tau^{\leq n} T_V M$. Then f_n induces an equivalence of categories $\mathcal{U} / \mathcal{K}_n \simeq \mathrm{Fun}_n$, where $\mathcal{K}_n$ denotes the Serre class consisting of all unstable $\mathcal{A}$-modules M such that $\tau^{\leq n} T_V M$ vanishes for every finite dimensional $\mathbf{F}_2$-vector space V.*

The following more precise description of $\mathcal{K}_n$ is available:

Theorem 13. *For each $n \geq 0$, the Serre class $\mathcal{K}_n \subseteq \mathcal{U}$ is the smallest Serre class containing $\Sigma^{n+1} M$, for every $M \in \mathcal{U}$.*

Proof. Since T_V commutes with suspension, we have

$$\tau^{\leq n} T_V \Sigma^{n+1} M \simeq \tau^{\leq n} \Sigma^{n+1} T_V M \simeq 0$$

for every $M \in \mathcal{U}$. This proves that $\Sigma^{n+1} M$ is contained in $\mathcal{K}_n$. The reverse inclusion is a nontrivial result which we will discuss in the next lecture. □

Example 14. The Serre classes $\mathrm{Nil}, \mathcal{K}_0 \subseteq \mathcal{U}$ coincide. The containment $\mathcal{K}_0 \subseteq \mathrm{Nil}$ is clear, since every suspension ΣM is nilpotent (in fact, the Frobenius map $\Phi M \to M$ is identically zero). Conversely, suppose that M is nilpotent. For each $k \geq 0$, let $M(k)$ denote the submodule of M consisting of elements x such that

$$\mathrm{Sq}^{2^k \deg(x)} \ldots \mathrm{Sq}^{2 \deg(x)} \mathrm{Sq}^{\deg x} x = 0.$$

Then $M = \bigcup_k M(k)$, so it will suffice to show that each $M(k) \in \mathcal{K}_0$. The proof then proceeds by induction on k. Since $\mathcal{K}_0$ is closed under extensions, it suffices to show that each $N = M(k)/M(k-1)$ belongs to $\mathcal{K}_0$. But the Frobenius map $\Phi N \to N$ is zero by construction, so the exact sequence

$$\Phi N \to N \to \Sigma \Omega N \to 0$$

proves that N is a suspension and therefore belongs to $\mathcal{K}_0$.