18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Krull Filtration (Lecture 37)

Let A be a commutative Noetherian ring. Recall that the *Zariski spectrum* $\operatorname{Spec} A$ is defined to be the set of all prime ideals $\{\mathfrak{p} \subseteq A\}$. Let Mod_A denote the category of A-modules. It is possible to recover $\operatorname{Spec} A$ directly from the category Mod_A. For this, we need to recall a few definitions and facts:

Definition 1. Let $\mathcal{C}$ be a Grothendieck abelian category. An object $X \in \mathcal{C}$ is *Noetherian* if every ascending chain of subobjects of X eventually stabilizes. We say that $\mathcal{C}$ is *locally Noetherian* if every object of $\mathcal{C}$ is the direct limit of its Noetherian subobjects.

An object $I \in \mathcal{C}$ is *injective* if the functor $M \mapsto \operatorname{Hom}_{\mathcal{C}}(M, I)$ is exact. We say that an injective object I is *indecomposable* if, whenever I is written as a direct sum $I \simeq I' \oplus I''$, either I' or I'' is zero.

Let $X \in \mathcal{C}$ be an object. An *injective hull* of X is a monomorphism $X \to I$ such that I is injective, and every nonzero subobject $I' \subseteq I$ satisfies $I' \times_I X \neq 0$.

Proposition 2. *Let $\mathcal{C}$ be a locally Noetherian abelian category. Then:*

(1) *Every object $M \in \mathcal{C}$ admits an injective hull $M \to I$. Moreover, I is uniquely determined up to (noncanonical) isomorphism. If M is simple, then I is indecomposable.*

(2) *Every direct sum $\oplus_\alpha I_\alpha$ of injective objects is injective.*

(3) *Every injective object $I \in \mathcal{C}$ can be obtained as a direct sum $\oplus_\alpha I_\alpha$, where each summand I_α is an indecomposable injective.*

This motivates the following definition:

Definition 3. Let $\mathcal{C}$ be a locally Noetherian abelian category. Then we let $\operatorname{Spec} \mathcal{C}$ denote the collection of all isomorphism classes of indecomposable injective objects of $\mathcal{C}$.

Remark 4. A priori, the collection $\operatorname{Spec} \mathcal{C}$ might be very large, since $\mathcal{C}$ has a proper class of injective objects. However, if I is an indecomposable injective object of $\mathcal{C}$, then I can be regarded as the injective hull of any nonzero submodule $I_0 \subseteq I$. In particular, I can be regarded as the injective hull of a Noetherian object of $\mathcal{C}$. It follows that $\operatorname{Spec} \mathcal{C}$ is actually a set.

Example 5. Let A be a Noetherian ring. Then there is a canonical bijection

$$\operatorname{Spec} A \to \operatorname{Spec} \operatorname{Mod}_A$$

which carries a prime ideal $\mathfrak{p} \subseteq A$ to the injective hull of the A-module $A/\mathfrak{p}$.

For example, if $A = \mathbf{Z}$, then the indecomposable injective objects of Mod_A are precisely the abelian groups $\mathbf{Q}$ and $\mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$, where p is a prime number.

Example 6. Let $\mathcal{U}$ denote the category of unstable Steenrod modules. The simple objects of $\mathcal{U}$ are precisely the modules $\Sigma^k \mathbf{F}_2$, where $k \geq 0$. The injective hull of $\Sigma^k \mathbf{F}_2$ can be identified with the Brown-Gitler module $J(k)$.

If A is a Noetherian ring, then $\operatorname{Spec} A$ has a good deal more structure than just that of a set. For example, we can (at least in good cases) assign a *Krull dimension* to every point of $\operatorname{Spec} A$. The points of Krull dimension zero correspond to the maximal ideals of A. Note that the collection of maximal ideals of A can be described very simply in terms of Mod_A: they are isomorphism classes of simple objects of Mod_A (more precisely, an A-module M is simple if and only if it is isomorphic to a quotient $A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of A). Therefore, the corresponding points of $\operatorname{Spec} \mathrm{Mod}_A$ are precisely the injective hulls of the simple objects of A. We now wish to generalize this picture to more general categories.

Definition 7. Let $\mathcal{C}$ be a locally Noetherian abelian category. Then $\mathrm{Krull}^0(\mathcal{C})$ is the smallest Serre class in $\mathcal{C}$ which contains every simple object in $\mathcal{C}$.

Remark 8. If $\mathcal{C} \neq 0$, then $\mathrm{Krull}^0(\mathcal{C}) \neq 0$. In other words, $\mathcal{C}$ contains a simple object. To prove this, choose a nonzero object $M \in \mathcal{C}$. Since $\mathcal{C}$ is locally Noetherian, M is the union of its Noetherian subobjects. We may therefore assume that M is Noetherian. Let M_0 be a maximal proper submodule of M. Then M/M_0 is a simple object of $\mathcal{C}$.

Proposition 9. *Let $\mathcal{C}$ be a locally Noetherian abelian category, and let I be an injective object of $\mathcal{C}$. Then exactly one of the following statements holds:*

(1) *The object I is the injective hull of a simple object $C \in \mathcal{C}$ (which is then determined up to isomorphism).*

(2) *The object I belongs to $\mathcal{C} / \mathrm{Krull}^0(\mathcal{C})$ (and is injective as an object of $\mathcal{C} / \mathrm{Krull}^0(\mathcal{C})$).*

Proof. Let $\mathcal{C}_0 = \{C \in \mathcal{C} : \mathrm{Hom}_{\mathcal{C}}(C, I) = 0\}$. Since I is injective, $\mathcal{C}_0$ is a Serre class in $\mathcal{C}$.

By definition, I belongs to $\mathcal{C} / \mathrm{Krull}^0(\mathcal{C})$ if and only if, for every object $C \in \mathrm{Krull}^0(\mathcal{C})$, we have $\mathrm{Hom}_{\mathcal{C}}(C, I) = \mathrm{Ext}_{\mathcal{C}}(C, I) = 0$. The second equality is automatic, since I is injective, and the first is equivalent to the assertion that $C \in \mathcal{C}_0$. In other words, $I \in \mathcal{C} / \mathrm{Krull}^0(\mathcal{C})$ if and only if $\mathrm{Krull}^0(\mathcal{C}) \subseteq \mathcal{C}_0$. Consequently, (2) holds if and only if $\mathrm{Hom}_{\mathcal{C}}(C, I) = 0$ for every simple object $C \in \mathcal{C}$.

Suppose that (2) does not hold, and choose a nonzero map $f : C \to I$ where C is simple. Then f must be a monomorphism. Choose an injective hull $C \subseteq I'$. Since I is injective, we can extend f to a map $\overline{f} : I' \to I$. Since $\ker(\overline{f}) \cap C \simeq \ker(f) \simeq 0$, we deduce that $\overline{f}$ is injective. Since I' is injective, the injective map $\overline{f}$ splits and we get an isomorphism $I \simeq I' \oplus I''$. Since I is indecomposable, $I'' \simeq 0$ so that $\overline{f}$ is an isomorphism. This proves (1), except for the uniqueness of C. To establish the uniqueness, we note that given injective maps

$$C \hookrightarrow I \hookleftarrow D,$$

the intersection $C \times_I D$ can be regarded as a nonzero submodule of both C and D. If C and D are simple, this gives isomorphisms

$$C \hookleftarrow C \times_I D \hookrightarrow D.$$

□

This motivates the following definition:

Definition 10. Let $\mathcal{C}$ be a Grothendieck abelian category. For each $n > 0$, we let $\mathrm{Krull}^n(\mathcal{C})$ denote the inverse image of $\mathrm{Krull}^0(\mathcal{C} / \mathrm{Krull}^{n-1}(\mathcal{C}))$ under the localization functor

$$L : \mathcal{C} \to \mathcal{C} / \mathrm{Krull}^{n-1}(\mathcal{C}).$$

We will say that an indecomposable injective $I \in \operatorname{Spec} \mathcal{C}$ has *Krull dimension* $> n$ if I belongs to $\mathcal{C} / \mathrm{Krull}^n \mathcal{C}$.

We have a filtration of $\mathcal{C}$ by Serre classes

$$\mathrm{Krull}^0(\mathcal{C}) \subseteq \mathrm{Krull}^1(\mathcal{C}) \subseteq \mathrm{Krull}^2(\mathcal{C}) \subseteq \ldots$$

By construction, each of the successive quotients $\mathrm{Krull}^{n+1}(\mathcal{C}) / \mathrm{Krull}^n(\mathcal{C})$ is generated by simple objects.

Remark 11. If A is a well-behaved commutative ring (such as a finitely generated algebra over a field), then the Krull filtration above is *finite*: we have $\mathrm{Krull}^n(\mathrm{Mod}_A) = \mathrm{Mod}_A$ as soon as $n \geq \dim(A)$. In general, the filtration need not terminate nor exhaust $\mathcal{C}$ (to obtain the whole of $\mathcal{C}$, one needs to define an analogous filtration indexed by the ordinals).

We wish to study the Krull filtration on the abelian category $\mathcal{U}$ of unstable $\mathcal{A}$-modules. We begin by determining $\mathrm{Krull}^0(\mathcal{A})$.

Definition 12. An unstable $\mathcal{A}$-module M is *locally finite* if, for each $x \in M$, the cyclic submodule $\mathcal{A}\,x \subseteq M$ has finite dimension over $\mathbf{F}_2$.

Proposition 13. *An unstable $\mathcal{A}$-module M belongs to* $\mathrm{Krull}^0(\mathcal{U})$ *if and only if M is locally finite.*

Proof. We first observe that the collection of locally finite $\mathcal{A}$-modules forms a Serre class in $\mathcal{U}$. Consequently, to prove the "only if" direction it will suffice to show that every simple $\mathcal{A}$-module is locally finite. This follows from the characterization of simple objects given in Remark **??**.

For the converse, let us suppose that M is locally finite. We wish to prove that $M \in \mathrm{Krull}^0(\mathcal{U})$. Write M as the union of its finitely generated submodules M_α. Since $\mathrm{Krull}^0(\mathcal{U})$ is a Serre class, it will suffice to show that each M_α belongs to $\mathrm{Krull}^0(\mathcal{U})$. Since M is locally finite, each M_α is finite dimensional over $\mathbf{F}_2$. We may therefore assume that M has finite dimension over $\mathbf{F}_2$. We now work by induction on the dimension of M. Let x be a nonzero element of M of maximal degree k. Then x determines an exact sequence

$$0 \to \Sigma^k \mathbf{F}_2 \to M \to M' \to 0.$$

By construction, we have $\Sigma^k \mathbf{F}_2 \in \mathrm{Krull}^0(\mathcal{U})$, and $M' \in \mathrm{Krull}^0(\mathcal{U})$ by the inductive hypothesis. It follows that $M \in \mathrm{Krull}^0(\mathcal{U})$, as desired. □

We now wish to give another characterization of $\mathrm{Krull}^0(\mathcal{U})$, this time using Lannes' T-functor. We first observe that $\mathrm{H}^*(B\mathbf{F}_2)$ canonically decomposes as a direct sum $\mathbf{F}_2 \oplus \mathrm{H}^*_{\mathrm{red}}(B\mathbf{F}_2)$. Consequently, we get a canonical isomorphism of functors

$$(\bullet \otimes \mathrm{H}^*(B\mathbf{F}_2)) \simeq \bullet \oplus (\bullet \otimes \mathrm{H}^*_{\mathrm{red}}(B\mathbf{F}_2)).$$

Passing to adjoints, we get a decomposition of functors

$$T \simeq \mathrm{id} \oplus \overline{T}$$

from the category $\mathcal{U}$ to itself. Moreover, formal properties of T are inherited by $\overline{T}$: for example, since T is exact and commutes with suspension and Φ, we deduce that $\overline{T}$ is exact and commutes with suspension and Φ.

Proposition 14. *Let M be an unstable $\mathcal{A}$-module. Then* $M \in \mathrm{Krull}^0(\mathcal{U})$ *if and only if* $\overline{T}M = 0$.

Proof. The "only if" direction is easy: let $\mathcal{C} = \{M \in \mathcal{U} : \overline{T}M = 0\}$. Then $\mathcal{C}$ is a Serre class in $\mathcal{U}$. To show that $\mathrm{Krull}^0(\mathcal{U}) \subseteq \mathcal{C}$, it suffices to show that every simple object $\Sigma^k \mathbf{F}_2$ belongs to $\mathcal{C}$. Since $\overline{T}$ commutes with suspensions, it suffices to show that $\overline{T}\mathbf{F}_2$ vanishes. This is equivalent to the assertion that $T\mathbf{F}_2 \simeq \mathbf{F}_2$, which was established in an earlier lecture.

The converse is much more difficult to prove. It relies on the following classification of the injective objects of $\mathcal{U}$:

Theorem 15. *Every indecomposable injective object of $\mathcal{U}$ appears as a summand of $J(m) \otimes (\mathrm{H}^*_{red}(B\mathbf{F}_2))^{\otimes n}$ for some integers m and n.*

Let us assume Theorem 15 and complete the proof. Let $M \in \mathcal{U}$ be such that $\overline{T}M = 0$. We wish to show that $M \in \mathrm{Krull}^0(\mathcal{U})$. Equivalently, we wish to show that the localization functor $L : \mathcal{U} \to \mathcal{U} / \mathrm{Krull}^0(\mathcal{U})$ annihilates M. If not, there exists a nonzero map $\eta \in \mathrm{Hom}(LM, I) \simeq \mathrm{Hom}(M, I)$, where I is an indecomposable injective of $\mathcal{U}\,/\,\mathrm{Krull}^0(\mathcal{U})$. According to Proposition 9, we can identify I with an indecomposable injective of $\mathcal{U}$ which is *not* the injective hull of a simple object (in other words, I is not isomorphic to a Brown-Gitler module $J(m)$). Invoking Theorem 15, we get a nonzero map

$$M \to J(m) \otimes \mathrm{H}^*_{\mathrm{red}}(B\mathbf{F}_2)^{\otimes n}$$

for some $n > 0$. This is adjoint to a nonzero map $\overline{T}^n M \to J(m)$, so that $\overline{T} M \neq 0$. □

We now extend the previous result to describe each step of the Krull filtration.

Proposition 16. *Let M be an unstable $\mathcal{A}$-module. Then $M \in \mathrm{Krull}^n(\mathcal{U})$ if and only if $\overline{T}^{n+1} M \simeq 0$.*

Proof. The proof goes by induction on n, the case $n = 0$ being Proposition 14. Suppose first that $\overline{T}^{n+1} M \simeq 0$. We wish to prove that $M \in \mathrm{Krull}^n(\mathcal{U})$. Writing M as the union of its finitely generated submodules, we may reduce to the case where M is finitely generated. Let $L : \mathcal{U} \to \mathcal{U}\,/\,\mathrm{Krull}^{n-1}(\mathcal{U})$ be the localization functor. We wish to show that LM belongs to $\mathrm{Krull}^0(\mathcal{U}\,/\,\mathrm{Krull}^{n-1}(\mathcal{U}))$. For this, we will show that LM has finite length in $\mathcal{U}\,/\,\mathrm{Krull}^{n-1}\,\mathcal{U}$.

By the inductive hypothesis, the functor $\overline{T}^n$ factors as a composition

$$\mathcal{U} \xrightarrow{L} \mathcal{U}\,/\,\mathrm{Krull}^{n-1}\,\mathcal{U} \xrightarrow{F} \mathcal{U}\,.$$

Consequently, for any subobject $N \subseteq LM$, we can identify FN with a subobject of $\overline{T}^n M$. Note that $\overline{T}^n M$ is locally finite (by Proposition 14) and finitely generated (since $\overline{T}$ preserves finitely generated objects), and therefore finite dimensional. Thus there are only finitely many possibilities for the subobject $FN \subseteq \overline{T}^n M$. But if $FN = FN' \subseteq \overline{T}^n M$, then the inclusions

$$N \hookleftarrow N \cap N' \hookrightarrow N'$$

induce isomorphisms

$$FN \hookleftarrow F(N \cap N') \hookrightarrow FN'.$$

Using the inductive hypothesis, we deduce that $N = N \cap N' = N'$. Thus, there are only finitely many subobjects of $LM \in \mathcal{U}\,/\,\mathrm{Krull}^{n-1}\,\mathcal{U}$, so that LM has finite length.

We now prove the reverse inclusion: $\mathrm{Krull}^n(\mathcal{U}) \subseteq \{M : \overline{T}^{n+1} M \simeq 0\}$. As before, the right side is a Serre class, to it will suffice to show that $\overline{T}^{n+1} M = 0$ whenever LM is a simple object of $\mathcal{U}\,/\,\mathrm{Krull}^{n-1}(\mathcal{U})$. We have a sequence of surjective maps

$$M \to \Sigma\Omega M \to \Sigma^2\Omega^2 M \to \dots$$

whose colimit is zero. Since LM is simple, we conclude that there exists an integer k such that the map

$$LM \to L\Sigma^k\Omega^k M$$

is an isomorphism and $L\Sigma^{k+1}\Omega^{k+1} M = 0$. We then have isomorphisms

$$\overline{T}^n M \to \overline{T}^n \Sigma^k \Omega^k M \simeq \Sigma^k \overline{T}^n \Omega^k M.$$

Moreover, the inductive hypothesis implies that Σ and Ω induce adjoint functors on the localized category $\mathcal{U}\,/\,\mathrm{Krull}^{n-1}(\mathcal{U})$; it is not difficult to deduce from this that $L\Omega^k M$ is again simple. We may therefore replace M by $\Omega^k M$, and thereby assume that $L\Sigma\Omega M \simeq 0$.

Consider the exact sequence

$$\Phi M \to M \to \Sigma\Omega M \to 0.$$

This gives rise to an exact sequence of localizations

$$L\Phi M \xrightarrow{\alpha} LM \to L\Sigma\Omega M \to 0$$

in the category $\mathcal{U} \,/\, \mathrm{Krull}^{n-1}(\mathcal{U})$. Since LM is simple and the last term vanishes, we conclude that α is an epimorphism.

Applying the functor F, we get an epimorphism $\overline{T}^n \Phi M \to \overline{T}^n M$. Let $N = \overline{T}^n M$. Since Φ commutes with $\overline{T}$, we deduce that the canonical map $\Phi N \to N$ is *surjective*. It then follows by induction on m that $N^m \simeq 0$ for $m > 0$. In other words, N is concentrated in degree zero, and is a direct sum of copies of $\mathbf{F}_2$. It follows that $0 \simeq \overline{T} N \simeq \overline{T}^{n+1} M$, as desired. $\square$