18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Free Modules (Lecture 7)

We first recall a bit of notation: If $I = (i_1, \ldots, i_k)$ is a sequence of integers, we write Sq^I for the composition product $\mathrm{Sq}^{i_1} \ldots \mathrm{Sq}^{i_k}$ in the Steenrod algebra $\mathcal{A}$ (or the big Steenrod algebra $\mathcal{A}^{\mathrm{Big}}$). We say that I is *admissible* if $i_j \geq 2i_{j+1}$ for $1 \leq j < k$. The *excess of* I is defined to be the expression

$$i_1 - i_2 - i_3 - \ldots - i_k = (i_1 - 2i_2) + (i_2 - 2i_3) + \ldots + (i_{k-1} - 2i_k) + i_k.$$

We wanted to prove that the Steenrod algebra has a basis $\{\mathrm{Sq}^I\}$, where I ranges over the admissible sequences of positive integers. This was reduced to the following assertion:

Proposition 1. *Let $F(n)$ denote the free unstable $\mathcal{A}$-module generated by one generator ν_n in degree n. Then the collection of elements $\{\mathrm{Sq}^I \nu_n\}$ is linearly independent in $F(n)$, where I ranges over admissible sequences of positive integers having excess $\leq n$.*

To prove this, it will suffice to find *any* unstable $\mathcal{A}$-module M with a element $x \in M^n$ such that the set $\{\mathrm{Sq}^I x\}$ is linearly independent in M (here again I ranges over admissible positive sequences of excess $\leq n$). To see this, we observe that the freeness of $F(n)$ implies that there is a (unique) map $\phi : F(n) \to M$ with $\phi(\nu_n) = x$. Consequently, any linear relation among the expressions $\{\mathrm{Sq}^I \nu_n\}$ would entail a linear relation among the expressions $\{\mathrm{Sq}^I x\}$.

It will therefore suffice to choose M to be some sufficiently nontrivial unstable $\mathcal{A}$-module. We have seen that for any topological space X, the cohomology $\mathrm{H}^*(X)$ has the structure of an unstable module over the Steenrod algebra. The most interesting example we have studied so far is the case where $X = B\Sigma_2 \simeq \mathbf{R}P^\infty$. In this case, the cohomology ring $\mathrm{H}^*(X)$ is isomorphic to a polynomial ring $\mathbf{F}_2[t]$, and the action of the Steenrod algebra is described by the formula

$$\mathrm{Sq}^k t^m = \binom{m}{k} t^{m+k}.$$

We can obtain a more interesting example by taking X to be a product of n copies of the space $\mathbf{R}P^\infty$. In this case, the cohomology of X can be identified with a polynomial ring $\mathbf{F}_2[t_1, \ldots, t_n]$ in several variables (obtained by pulling back the cohomology class t along the n different projections). Using the Cartan formula

$$\mathrm{Sq}^k(xy) = \sum_{k=k'+k''} \mathrm{Sq}^{k'}(x) \, \mathrm{Sq}^{k''}(y),$$

we deduce that the action of the Steenrod algebra on $\mathrm{H}^*(X)$ is described by the following formula:

$$\mathrm{Sq}^k(t_1^{a_1} \ldots t_n^{a_n}) = \sum_{k=k_1+\ldots+k_n} \binom{a_1}{k_1} \ldots \binom{a_n}{k_n} t_1^{a_1+k_1} \ldots t_n^{a_n+k_n}.$$

We now make a crucial observation about the formula above. Suppose that each exponent a_i is a power of 2. The binomial coefficient $\binom{a_i}{k_i}$ is equal to 1 if $k_i = 0$ or $k_i = a_i$, and vanishes otherwise (since we are working over the field $\mathbf{F}_2$). Moreover, the exponents appearing on the right hand side have the form $a_i + k_i$, which will again be a power of two if $k_i = 0$ or $k_i = a_i$. In other words, we can rewrite the preceding formula as follows:

$$\mathrm{Sq}^k(t_1^{2^{b_1}} \dots t_n^{2^{b_n}}) = \sum_{k=\delta_1 2^{b_1}+\dots+\delta_n 2^{b_n}} t_1^{2^{b_1+\delta_1}} \dots t_n^{2^{b_n+\delta_n}},$$

where the sum is taken over $\delta_1, \dots, \delta_n \in \{0, 1\}$.

Let $x = t_1 \dots t_n \in \mathbf{F}_2[t_1, \dots, t_n]$. Then, for every sequence of integers I, the expression $\mathrm{Sq}^I(x)$ can be identified with some polynomial $f(t_1, \dots, t_n) \in \mathbf{F}_2[t_1, \dots, t_n]$. This polynomial necessarily has the following properties:

(a) Every monomial appearing in f has the form $t_1^{2^{b_1}} \dots t_n^{2^{b_n}}$.

(b) The polynomial f is symmetric in its arguments.

Let M denote the subspace of $\mathbf{F}_2[t_1, \dots, t_n]$ consisting of those polynomials which satisfy (a) and (b) above. We observe that M is invariant under the action of the Steenrod algebra $\mathcal{A}$, and is therefore an unstable $\mathcal{A}$-module in its own right. Moreover, M contains the element $x = t_1 \dots t_n$ of degree n. To complete the proof of Proposition 1, it will suffice to show the following:

Proposition 2. *The expressions $\{\mathrm{Sq}^I(x)\}$ form a basis for M, where I ranges over admissible sequences of positive integers having excess $\leq n$.*

Let us now introduce a bit of notation. Given a monomial $f = t_1^{a_1} \dots t_n^{a_n}$, let

$$\sigma(f) = \sum_{g \in \Sigma_n / G} f^g$$

be the symmetric polynomial obtained by summing the conjugates of f; here we take G to be the stabilizer of f in Σ_n, so that f itself appears in this sum exactly once. For example, if $n = 2$, we have

$$\sigma(t_1^a t_2^b) = \begin{cases} t_1^a t_2^b & \text{if } a = b \\ t_1^a t_2^b + t_1^b t_2^a & \text{if } a \neq b \end{cases}.$$

The space M has a basis consisting of symmetric polynomials of the form $\sigma(t_1^{2^{b_1}} \dots t_n^{2^{b_n}})$, where $0 \leq b_1 \leq \dots \leq b_n$. It will be convenient to index this set of polynomials a little bit differently. Given a sequence of nonnegative integers $\epsilon = (\epsilon_0, \dots, \epsilon_k)$ with $\epsilon_0 + \dots + \epsilon_k = n$, there is a unique sequence $0 \leq b_1 \leq \dots \leq b_n$ such that ϵ_i is the cardinality of the set $\{j : b_j = i\}$. We then set $f_\epsilon = \sigma(t_1^{2^{b_1}} \dots t_n^{2^{b_n}})$. Thus M has a basis consisting of the polynomials $\{f_\epsilon\}$, where ϵ ranges over sequences of nonnegative integers $(\epsilon_0, \dots, \epsilon_k)$ such that $n = \epsilon_0 + \dots + \epsilon_k$ and ϵ_k is nonzero.

There is a corresponding indexing for positive admissible monomials of the form Sq^I. Let $I = (i_1, \dots, i_k)$ be a sequence of positive integers. If I is admissible, then the integers $\epsilon_1 = i_1 - 2i_2$, $\epsilon_2 = i_2 - 2i_3, \dots, \epsilon_{k-1} = i_{k-1} - 2i_k$ are all nonnegative. We then set $\epsilon_k = i_k$, which is positive so long as I is positive. The sum

$$\epsilon_1 + \dots + \epsilon_k = i_1 - i_2 - \dots - i_k$$

is equal to the excess of I. Thus, if I has excess $\leq n$, we can define $\epsilon_0 = n - (\epsilon_1 + \dots + \epsilon_k)$, to obtain a sequence of nonnegative integers $\epsilon = (\epsilon_0, \dots, \epsilon_k)$, where ϵ_k is positive. Conversely, given such a sequence of integers, we can construct a unique admissible sequence $I = (2^{k-1}\epsilon_k + \dots + \epsilon_1, \dots, 2\epsilon_k + \epsilon_{k-1}, \epsilon_k)$ of excess $\leq n$. We will denote this admissible sequence by $I(\epsilon)$.

We now wish to compare the expressions $\{\mathrm{Sq}^{I(\epsilon)}(x)\}$ with the basis $\{f_\epsilon\}$ for M. They do not coincide, but we get the next best thing: the translation between these two bases is upper triangular. To be more precise, we need to introduce an ordering on our index set. Let E be the collection of all finite sequences $\epsilon = (\epsilon_0, \dots, \epsilon_k)$ of nonnegative integers (here k is allowed to vary) such that $\epsilon_k > 0$, and $\epsilon_0 + \dots + \epsilon_k = n$.

We equip E with the following lexicographical ordering: $\epsilon < \epsilon'$ if there exists an integer i such that $\epsilon_i < \epsilon'_i$, while $\epsilon_j = \epsilon'_j$ for $j > i$. Here we agree to the convention that $\epsilon_i = 0$ if i is larger than the length of the sequence ϵ.

To complete prove Proposition 2, it will suffice to verify the following:

Proposition 3. *Let* $\epsilon \in E$. *Then*

$$\mathrm{Sq}^{I(\epsilon)}(x) = f_\epsilon + \sum_\alpha f_\alpha$$

where α *ranges over some subset of* $\{\epsilon' \in E : \epsilon' < \epsilon\}$.

Proof. We compute:

$$x = \sigma(t_1 \dots t_n)$$

$$\mathrm{Sq}^{\epsilon_k}(x) = \sigma(t_1^2 t_2^2 \dots t_{\epsilon_k}^2 t_{\epsilon_k+1} \dots t_n)$$

$$\mathrm{Sq}^{\epsilon_{k-1}+2\epsilon_k}\, \mathrm{Sq}^{\epsilon_k}(x) = \sigma(t_1^4 t_2^4 \dots t_{\epsilon_k}^4 t_{\epsilon_k+1}^2 \dots t_{\epsilon_k+\epsilon_{k-1}}^2 t_{\epsilon_k+\epsilon_{k-1}+1} \dots t_n) + \text{lower order}$$

$$\dots$$

$$\mathrm{Sq}^{I(\epsilon)}(x) = f_\epsilon + \text{lower order}$$

□

We now wish to reformulate some of the above ideas, using Kuhn's theory of "generic representations". In what follows, we let V denote a finite dimensional vector space over $\mathbf{F}_2$, and let $V^\vee$ denote its dual space. We observe that

$$\mathrm{H}^*(BV^\vee) = \mathrm{H}^*(\mathbf{R}P^\infty \times \dots \times \mathbf{R}P^\infty) \simeq \mathbf{F}_2[t_1, \dots, t_N],$$

where N is the dimension of V. However, we can describe this cohomology ring more in a more invariant way: it is given by the symmetric algebra $\mathrm{Sym}^*(V)$ generated by the vector space $V \simeq \mathrm{H}^1(BV^\vee)$.

Every admissible monomial Sq^I in the Steenrod algebra of degree k determines a map

$$\mathrm{H}^*(BV^\vee) \to \mathrm{H}^{*+k}(BV^\vee).$$

Restricting to a particular degree n, we get a map

$$\mathrm{Sym}^n(V) \to \mathrm{Sym}^{n+k}(V).$$

This map depends functorially on V, and vanishes if the excess of I is larger than n.

To study the situation more systematically, let Vect^f denote the category of finite dimensional vector spaces over $\mathbf{F}_2$, and Vect the category of *all* vector spaces over $\mathbf{F}_2$. We let Fun denote the category of functors from Vect^f to Vect.

Remark 4. Kuhn refers to objects of Fun as *generic representations*. If $F : \mathrm{Vect}^f \to \mathrm{Vect}$ is a functor, then for every finite dimensional vector space $V \in \mathrm{Vect}^f$, we obtain a new vector space $F(V)$ which is equipped with an action of $\mathrm{Aut}(V) \simeq \mathrm{GL}_n(\mathbf{F}_2)$. In other words, we can think of F as providing a family of representations of the groups $\mathrm{GL}_n(\mathbf{F}_2)$, which are somehow connected to one another as n grows.

Example 5. For every nonnegative integer n, the functor

$$V \mapsto \mathrm{Sym}^n(V)$$

is an object of Fun, which we will denote by Sym^n. Let Sym^* denote the direct sum of these functors, so that $\mathrm{Sym}^*(V)$ is the free algebra generated by V.

If Sq^I is an admissible monomial (or any element of the Steenrod algebra), then Sq^I determines a natural transformation

$$\mathrm{Sym}^n \to \mathrm{Sym}^*;$$

in other words, a morphism in the category Fun. This natural transformation vanishes if the excess of I is larger than n.

Proposition 6. *Let n be a positive integer. Then the natural transformations $\{\mathrm{Sq}^I\}$ form a basis for $\mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^*)$, where I ranges over positive admissible sequences of excess $\leq n$.*

Proof. We first show that the expressions Sq^I are linearly independent in $\mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^*)$. For this, it suffices to choose a vector space V such that the functors Sq^I are linearly independent in $\mathrm{Hom}_{\mathbf{F}_2}(\mathrm{Sym}^n(V), \mathrm{Sym}^*(V))$. Let V be the free vector space generated by a basis $\{t_1, \ldots, t_n\}$, and let $x = t_1 \ldots t_n$; then it will suffice to show that the elements $\{\mathrm{Sq}^I(x)\}$ are linearly independent in $\mathrm{Sym}^*(V)$. This follows immediately from Proposition 2.

We now wish to prove that $\mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^*)$ is spanned by the Steenrod operations $\{\mathrm{Sq}^I\}$. For this, we need to compute $\mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^*)$. Suppose $\alpha : \mathrm{Sym}^n \to \mathrm{Sym}^*$ is a natural transformation. Choose $V = \mathbf{F}_2\{t_1, \ldots, t_n\}$ as above, and let $x = t_1 \ldots t_n \in \mathrm{Sym}^n(V)$. Then $\alpha(x) = f(t_1, \ldots, t_n) \in \mathbf{F}_2[t_1, \ldots, t_n] \simeq \mathrm{Sym}^*(V)$, for some polynomial f. The construciton $\alpha \mapsto f$ determines a linear map

$$\phi : \mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^*) \to \mathbf{F}_2[t_1, \ldots, t_n].$$

We first claim that ϕ is injective. For suppose that $\phi(\alpha) = 0$. Let W be any vector space over $\mathbf{F}_2$. We wish to prove that the induced map

$$\alpha_W : \mathrm{Sym}^n(W) \to \mathrm{Sym}^*(W)$$

is equal to zero. Since α_W is a linear map, it will suffice to show that α_W vanishes on each monomial $w_1 \ldots w_n$ in $\mathrm{Sym}^n(W)$. But in this case we have a map $V \to W$, given by $t_i \mapsto w_i$. This linear map determines a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^n(V) & \xrightarrow{\phi} & \mathrm{Sym}^*(V) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^n(W) & \xrightarrow{\alpha_W} & \mathrm{Sym}^*(W), \end{array}$$

so that $\alpha_W(w_1 \ldots w_n) = f(w_1, \ldots, w_n) = 0 \in \mathrm{Sym}^*(W)$.

We now wish to describe the image of the map ϕ. Fix $\alpha : \mathrm{Sym}^n \to \mathrm{Sym}^*$, and let $f = \phi(\alpha)$. Since $x = t_1 \ldots t_n \in \mathrm{Sym}^n(V)$ is invariant under the permutation action of the symmetric group, we deduce immediately that f is a *symmetric* polynomial.

Let V' be the $\mathbf{F}_2$-vector space spanned by a basis $\{t_1, \ldots, t_n, t_{n+1}\}$. Then we have an equation

$$t_1 \ldots t_{n-1}(t_n + t_{n+1}) = t_1 \ldots t_n + t_1 \ldots t_{n-1} t_{n+1}.$$

Since the map $\alpha_{V'}$ is linear, we get

$$f(t_1, \ldots, t_{n-1}, t_n + t_{n+1}) = f(t_1, \ldots, t_n) + f(t_1, \ldots, t_{n-1}, t_{n+1}).$$

In other words, the polynomial f is *additive* in its last argument. If we write

$$f(t_1, \ldots, t_n) = \sum_k g_k(t_1, \ldots, t_{n-1}) t_n^k,$$

then we deduce that $g_k(t_1, \ldots, t_{n-1})$ vanishes unless k is a power of 2. Since f is symmetric, we can apply the same reasoning to each argument of f. It follows that f can be written as a sum of monomials of the form $t_1^{2^{b_1}} \ldots t_n^{2^{b_n}}$. Since f is symmetric, we conclude that $f \in M \subseteq \mathbf{F}_2[t_1, \ldots, t_n]$.

We therefore have a factorization

$$\phi : \mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^*) \hookrightarrow M \subseteq \mathbf{F}_2[t_1, \ldots, t_n].$$

The map ϕ carries Sq^I to $\mathrm{Sq}^I(x)$. Proposition 2 implies that M is generated by these expressions, so that ϕ restricts to an isomorphism $\mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^*) \simeq M$. Since the expressions $\{\mathrm{Sq}^I(x)\}$ form a basis for M (where I ranges over admissible positive sequences of excess $\leq n$), we conclude that the expressions $\{\mathrm{Sq}^I\}$ form a basis for $\mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^*)$. □

This gives another approach to constructing the Steenrod algebra (at least with mod-2 coefficients): it can be regarded as an algebra of natural transformations between functors of the form

$$\mathrm{Sym}^n : \mathrm{Vect}^f \to \mathrm{Vect}\,.$$

We will return to this point of view in the next lecture.