18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

A Theorem of Gabriel-Kuhn-Popesco (Lecture 8)

Let $\mathcal{C}$ be an abelian category. Suppose that $\mathcal{C}$ is equivalent to the category $\mathrm{Mod}(R)$ of (right) modules over an associative ring R. How might we recognize this? To answer this, we first recall a very general definition:

Definition 1. Let $\mathcal{C}$ be an abelian category which admits direct sums. A collection of objects $\{C_\alpha\}$ *generates* $\mathcal{C}$ if, for every object $D \in \mathcal{C}$, there exists an epimorphism

$$\oplus_i C_{\alpha_i} \to D.$$

A *Grothendieck abelian category* is an abelian category $\mathcal{C}$ satisfying the following conditions:

(1) The category $\mathcal{C}$ admits filtered colimits, and the formation of filtered colimits is exact (in other words, a filtered colimit of monomorphisms is a monomorphism).

(2) There exists a set of generators $\{C_\alpha\}$ for $\mathcal{C}$.

If $\mathcal{C}$ is equivalent to $\mathrm{Mod}(R)$, then it has a distinguished object C, corresponding to R (regarded as a module over itself). We can then recover R as the ring of endomorphisms $\mathrm{Hom}_{\mathcal{C}}(C, C)$. More generally, given any object $D \in \mathcal{C}$, we can define a (right) R-module $G(D)$ by the formula

$$G(D) = \mathrm{Hom}_{\mathcal{C}}(C, D).$$

If $\mathcal{C}$ is a *Grothendieck* abelian category, then the functor G has a left adjoint F, which we will denote by

$$M \mapsto M \otimes_R C.$$

The adjoint functors F and G determine an equivalence between $\mathcal{C}$ and Mod_R if and only if the following three conditions are satisfied:

(1) The object C generates $\mathcal{C}$.

(2) The object C is projective: that is, the functor $\mathrm{Hom}_{\mathcal{C}}(C, \bullet)$ is exact.

(3) The object C is compact: that is, the functor $\mathrm{Hom}_{\mathcal{C}}(C, \bullet)$ commutes with filtered colimits (in view of (2), this is equivalent to requiring that $\mathrm{Hom}_{\mathcal{C}}(C, \bullet)$ commutes with all colimits, or with direct sums).

If C fails to satisfy conditions (2) and (3), then there is still a close relationship between $\mathcal{C}$ and $\mathrm{Mod}(R)$: namely, $\mathcal{C}$ is a *localization* of $\mathrm{Mod}(R)$. This is the classical *Gabriel-Popesco theorem*.

Condition (1) is not very restrictive: every Grothendieck abelian category admits a generator. Note, for example, that if $\mathcal{C}$ is generated by a *set* of objects $\{C_\alpha\}$, then $\mathcal{C}$ is generated by the single object $C = \oplus_\alpha C_\alpha$. However, the ring $R = \mathrm{Hom}_{\mathcal{C}}(C, C)$ in this case might be rather unwieldy. It will therefore be convenient to formulate a "many-object" version of the Gabriel-Popesco theorem. We will follow the presentation of Nick Kuhn.

Throughout the remainder of this lecture, we fix the following notation:

- $\mathcal{C}$ will be a Grothendieck abelian category.
- $\{C_\alpha\}$ will be a set of objects of $\mathcal{C}$ which generates $\mathcal{C}$.
- $\mathcal{R}$ will denote the full subcategory of $\mathcal{C}$ spanned by the objects $\{C_\alpha\}$.

Definition 2. A $\mathcal{R}$-*module* is a contravariant functor M from $\mathcal{R}$ to the category of abelian groups, which is linear in the following sense: for every pair of objects $C, D \in \mathcal{R}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(C, D) \times M(D) \to M(C)$$

is bilinear.

The collection of $\mathcal{R}$-modules can be organized into a category, which we will denote by $\mathrm{Mod}(\mathcal{R})$.

Example 3. If $\mathcal{R}$ consists of a single object $C \in \mathcal{C}$, then a $\mathcal{R}$-module is simply a right module over the ring $R = \mathrm{Hom}_{\mathcal{C}}(C, C)$.

Example 4. Let D be an object of $\mathcal{C}$. Then the functor $C \mapsto \mathrm{Hom}_{\mathcal{C}}(C, D)$ is a $\mathcal{R}$-module. We will denote this $\mathcal{R}$-module by $G(D)$. This construction determines a functor

$$G : \mathcal{C} \to \mathrm{Mod}(\mathcal{R}).$$

Theorem 5 (Kuhn, Gabriel-Popesco)**.** (1) *The functor G admits a left adjoint F.*

(2) *The functor G is fully faithful.*

(3) *The functor F is exact.*

Remark 6. Theorem 5 implies that $\mathcal{C}$ can be obtained as a *localization* of $\mathrm{Mod}(\mathcal{R})$. More precisely, let $\mathcal{K}$ denote the full subcategory of $\mathrm{Mod}(\mathcal{R})$ spanned by those modules M such that $F(M) \simeq 0$. Then $\mathcal{K}$ is a Serre subcategory of $\mathrm{Mod}(\mathcal{R})$, and F induces an equivalence

$$\mathrm{Mod}(\mathcal{R})/\,\mathcal{K} \simeq \mathcal{C}\,.$$

The rest of this lecture is devoted to proving Theorem 5. We will later apply this theorem in the case where $\mathcal{C}$ is the category $\mathrm{Fun} = \mathrm{Fun}(\mathrm{Vect}^f, \mathrm{Vect})$. Combined with the results of the previous lecture, this will yield some interesting information on the category of unstable modules over the Steenrod algebra $\mathcal{A}$.

Assertion (1) follows from the adjoint functor theorem. To prove (2) and (3) we will follow the argument presented in Kuhn, "Generic Representations of the Finite General Linear GRoups and the Steenrod Algebra I".

Lemma 7. *Let M be an $\mathcal{R}$-module and let $D \in \mathcal{C}$. If $u : M \to G(D)$ is a monomorphism in* $\mathrm{Mod}(\mathcal{R})$*, then the adjoint map $u' : F(M) \to D$ is a monomorphism in $\mathcal{C}$.*

Proof. We first observe that there is an epimorphism

$$\pi : \oplus_{\alpha \in M(C)} C \to F(M).$$

To prove that u' is a monomorphism, it will suffice to show that $\ker(u' \circ \pi) = \ker(\pi)$. Since $\mathcal{R}$ generates $\mathcal{C}$,

The direct sum $\oplus_{\alpha \in M(C)} C$ is a direct limit of finite sums

$$\oplus_{i \in I} C_i$$

Let π_I denote the restriction of π to this finite sum. Since filtered colimits in $\mathcal{C}$ are exact, we deduce that

$$\ker(u' \circ \pi) \simeq \operatorname{colim} \ker(u' \circ \pi_I)$$

$$\ker(\pi) \simeq \operatorname{colim} \ker(\pi_I).$$

It will therefore suffice to show that $\ker(u' \circ \pi_I) = \ker(\pi_I)$ for every finite set I.

Since $\mathcal{R}$ generates $\mathcal{C}$, it will suffice to show that for every, $C \in \mathcal{R}$, any map $C \to \ker(u' \circ \pi_I)$ factors through $\ker(\pi_I)$. In other words, we must show that if we are given a diagram

$$C \xrightarrow{\beta} \oplus_{i \in I} C_i \xrightarrow{\pi_I} F(M) \xrightarrow{u'} D$$

such that $u' \circ \pi_I \circ \beta = 0$, then $\pi_i \circ \beta = 0$. The map β corresponds to a family of maps $\{\beta_i : C \to C_i\}_{i \in I}$, and the map π_I is given by a family of elements $\{\alpha_i \in M(C_i)\}_{i \in I}$. We now observe that $\pi_I \circ \beta$ is the map given by

$$\gamma = \sum_{i \in I} \alpha_i \beta_i \in M(C).$$

The map $u' \circ \pi_I \circ \beta$ can be identified with $u(\gamma) \in G(D)(C) \simeq \operatorname{Hom}_{\mathcal{C}}(C, D)$. Since the map u is a monomorphism, the equation $u' \circ \pi_I \circ \beta = 0$ implies $\gamma = 0$, so that $\pi_I \circ \beta$ also vanishes. □

Corollary 8. *Let $C \in \mathcal{C}$. The the counit map $v : FG(C) \to C$ is an isomorphism.*

Proof. The counit map is adjoint to the isomorphism $G(C) \to G(C)$. Lemma 7 implies that v is a monomorphism.

Let $C' \in \mathcal{R}$, and let $\alpha : C' \to C$ be a morphism in $\mathcal{C}$. Then α can be viewed as an element of $G(C)(C')$, and therefore determines a map $\alpha' : C' \to FG(C)$ such that $v \circ \alpha' = \alpha$. In other words, every map $C' \to C$ factors through v if $C' \in \mathcal{R}$. Since $\mathcal{R}$ generates $\mathcal{C}$, we deduce that v is an epimorphism. □

Corollary 9. *The functor G is fully faithful.*

Proof. For every pair of objects $C, D \in \mathcal{C}$, we have isomorphisms

$$\operatorname{Hom}_{\mathcal{C}}(C, D) \simeq \operatorname{Hom}_{\mathcal{C}}(FG(C), D) \simeq \operatorname{Hom}_{\operatorname{Mod}(\mathcal{R})}(G(C), G(D)).$$

□

Let us say that an object $M \in \operatorname{Mod}(\mathcal{R})$ is *free* if it is a direct sum of objects of the form $G(C)$, where $C \in \mathcal{R}$. For any $\mathcal{R}$-module N, Yoneda's lemma yields an isomorphism

$$\operatorname{Hom}_{\operatorname{Mod}(\mathcal{R})}(G(C), N) = N(C).$$

Since the evaluation functors $N \mapsto N(C)$ are exact, we conclude that the free objects of $\operatorname{Mod}(\mathcal{R})$ are *projective*. Moreover, $\operatorname{Mod}(\mathcal{R})$ is generated by free objects: for any $N \in \operatorname{Mod}(\mathcal{R})$, the map

$$\oplus_{\alpha \in N(C)} G(C) \to N$$

is an epimorphism. Consequently, every $N \in \operatorname{Mod}(\mathcal{R})$ admits a free resolution

$$\ldots \to P_1 \to P_0 \to N.$$

We can therefore define the left derived functors of F: by definition, $L^i F(N)$ is the ith homology of the complex

$$\ldots \to F(P_1) \to F(P_0).$$

Since the functor F preserves colimits, we deduce that

$$L^0 F(N) \simeq \operatorname{coker}(F(P_1) \to F(P_0)) \simeq F \operatorname{coker}(P_1 \to P_0) \simeq F(N).$$

That is, F is its own 0th derived functor.

For every short exact sequence of $\mathcal{R}$-modules

$$0 \to M' \to M \to M'' \to 0,$$

we get a long exact sequence of right derived functors

$$\ldots \to L^1 F(M'') \to F(M') \to F(M) \to F(M'') \to 0.$$

Consequently, to prove that F is exact it will suffice to show that the derived functors $L^i F$ vanish for $i > 0$. In other words, it will suffice to show:

Lemma 10. *Suppose given an exact sequence of $\mathcal{R}$-modules*

$$\ldots \to P_1 \to P_0 \to N,$$

where each P_i is free. Then the induced sequence

$$\ldots \to F(P_1) \to F(P_0) \to F(N)$$

is exact in $\mathcal{C}$.

To prove Lemma 10, we note that a long exact sequence is obtained by can be obtained by splicing together short exact sequences

$$0 \to \mathrm{Im}(P_1 \to P_0) \to P_0 \to N$$

$$0 \to \mathrm{Im}(P_2 \to P_1) \to P_1 \to \mathrm{Im}(P_1 \to P_0) \to 0$$

$$\ldots$$

It will suffice to show that the functor F preserves each of these short exact sequences. Since F preserves colimits, it is automatically right exact. So the only question is whether or not F preserves the monic arrows which appear above. This follows from:

Lemma 11. *Let P be a free $\mathcal{R}$-module, and let $M \subseteq P$. Then the induced map $F(M) \to F(P)$ is a monomorphism in $\mathcal{C}$.*

Proof. We can write $P = \mathrm{colim}\{P_\alpha\}$, where each P_α is a *finitely generated* free module. Let $M_\alpha = M \cap P_\alpha$. Then the map $F(M) \to F(P)$ is a filtered colimit of maps of the form $F(M_\alpha) \to F(P_\alpha)$. Since the collection of monomorphisms in $\mathcal{C}$ is stable under filtered colimits, we may reduce to the case where $P = P_\alpha$ is finitely generated.

In this case, we can choose a finite collection of objects $\{C_i \in \mathcal{R}\}_{1 \leq i \leq n}$ such that $P = \oplus_{1 \leq i \leq n} G(C_i)$. Let $C = \oplus_{1 \leq i \leq n} C_i$, so that $P = G(C)$. Then

$$F(P) \simeq \oplus_{1 \leq i \leq n} FG(C_i) \simeq \oplus_{1 \leq i \leq n} C_i \simeq C.$$

The map $F(M) \to F(P) \simeq C$ is adjoint to the inclusion $M \subseteq P \simeq G(C)$, and is therefore a monomorphism by Lemma 7. □