18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Injectivity of $H^*(BV)$ (Lecture 9)

Let n be a nonnegative integer, and let Sq^I be an element of the Steenrod algebra $\mathcal{A}$. We have seen that Sq^I determines a map $\mathrm{Sym}^n \to \mathrm{Sym}^*$ in the category $\mathrm{Fun} = \mathrm{Fun}(\mathrm{Vect}^f, \mathrm{Vect})$, where Vect is the category of $\mathbf{F}_2$-vector spaces and $\mathrm{Vect}^f \subseteq \mathrm{Vect}$ is the category of finite dimensional $\mathbf{F}_2$-vector spaces. If we keep track of degrees, we can be more precise: Sq^I determines a map $\mathrm{Sq}^n \to \mathrm{Sq}^{n+\deg(I)}$. This map vanishes if the excess of I is larger than n. Moreover, we proved:

Proposition 1. *Let m and n be nonnegative integers. Then $\mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^m)$ has a basis given by the Steenrod operations $\{\mathrm{Sq}^I\}$, where I ranges over positive admissible sequences of degree $m-n$ and excess $\leq n$.*

In particular, there are no nontrivial natural transformations from Sym^n to Sym^m for $m < n$.

We can express Proposition 1 in a slightly different way. Let $F(n)$ denote the free unstable $\mathcal{A}$-module on a single generator ν_n. Then the expressions $\{\mathrm{Sq}^I \nu_n\}$ form a basis for $F(n)$, where I ranges over the collection of positive admissible sequences of excess $\leq n$. If we restrict our attention to admissible sequences of degree $m-n$, then we get a basis for the mth graded piece $F(n)^m \simeq \mathrm{Hom}_{\mathcal{A}}(F(m), F(n))$. We may therefore reformulate Proposition 1 as follows:

Proposition 2. *Let m and n be nonnegative integers. Then there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Sym}^n, \mathrm{Sym}^m) \simeq \mathrm{Hom}_{\mathcal{A}}(F(m), F(n)).$$

Let $\mathcal{U}$ denote the category of unstable modules over the Steenrod algebra. Unwinding the definitions, we see that the isomorphism of Proposition 2 is compatible with composition. It therefore defines an *anti-equivalence* between the full subcategory of Fun spanned by the objects $\{\mathrm{Sym}^n\}_{n\geq 0}$ and the full subcategory of $\mathcal{U}$ spanned by the objects $\{F(n)\}_{n\geq 0}$. We wish to apply the results of the last lecture to this situation. First, we need to convert the anti-equivalence of Proposition 2 into a covariant equivalence.

Definition 3. Let $F : \mathrm{Vect}^f \to \mathrm{Vect}$ be a functor. We let DF denote the functor

$$F \mapsto F(V^\vee)^\vee,$$

where $V^\vee$ denotes the vector space dual to $\vee$. We will refer to DF as the *dual* to F.

We note that DF is again a covariant functor from Vect^f to Vect, and the the construction

$$F \mapsto DF$$

determines a contravariant functor from Fun to itself. Moreover, for every functor F there is a canonical map $F \mapsto DDF$, which is an isomorphism if and only if each of the vector spaces $F(V)$ is finite-dimensional. It follows that if F and G take values in finite dimensional vector spaces, then we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}}(F, G) \simeq \mathrm{Hom}_{\mathrm{Fun}}(DG, DF).$$

(In fact, we have such an isomorphism whenever the values of G are finite-dimensional.)

Example 4. For each $n \geq 0$, we let $\Gamma^n : \mathrm{Vect}^f \to \mathrm{Vect}$ denote the functor

$$V \mapsto (V^{\otimes n})^{\Sigma_n}.$$

Then Γ^n is isomorphic to the dual $D\,\mathrm{Sym}^n$.

We can reformulate Proposition 2 as follows:

Proposition 5. *Let m and n be nonnegative integers. Then there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{Fun}}(\Gamma^m, \Gamma^n) \simeq \mathrm{Hom}_{\mathcal{A}}(F(m), F(n)).$$

Let $\mathcal{R}$ denote the full subcategory of Fun spanned by the functors $\{\Gamma^n\}_{n\geq 0}$. We would like to apply Kuhn's many-object version of the Gabriel-Popesco theorem to the subcategory $\mathcal{R} \subseteq \mathrm{Fun}$. Unforunately, the hypotheses of the theorem are not satisfied: the category Fun is not generated by the objects $\{\Gamma^n\}_{n\geq 0}$. We can remedy the situation by passing to a suitable subcategory of Fun.

Definition 6. For every functor $F \in \mathrm{Fun}$, we define a new functor $\Delta(F)$ by the formula

$$\Delta(F)(V) = \ker(F(V \oplus \mathbf{F}_2) \to F(V)).$$

We say that a functor $F \in \mathrm{Fun}$ is *polynomial of degree* $\leq n$ if $\Delta^{n+1}(F)$ vanishes.

We observe that for any functor F, we have a canonical splitting

$$F(V \oplus \mathbf{F}_2) \simeq F(V) \oplus \Delta(F)(V).$$

It follows the functor $F \mapsto \Delta(F)$ is exact. Moreover, it is clear that the functor Δ commutes with infinite direct sums. From this we immediately deduce:

Lemma 7. *The collection of polynomial functors of degree $\leq n$ is closed under the formation of subobjects, quotient objects, and extensions in the category* Fun.

Remark 8. Let $F \in \mathrm{Fun}$ be a functor which takes values in finite dimensional vector space, and let $d_F : \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ be the function defined by the formula

$$d_F(n) = \dim F(\mathbf{F}_2^n).$$

We note that $d_{\Delta(F)}(n) = d_F(n+1) - d_F(n)$, and that F vanishes if and only if d_F vanishes. It follows that F is polynomial of degree $\leq n$ if and only if the function d_F is a polynomial of degree $\leq n$.

Example 9. Let $n \geq 0$. Then

$$d_{\mathrm{Sym}^n}(k) = d_{\Gamma^n}(k) = \binom{n+k-1}{n}.$$

Consequently, the functors Sym^n and Γ^n are polynomial of degree exactly n.

Lemma 10. *For every functor $F \in$* Fun, *there exists a maximal subfunctor $F^{(n)} \subseteq F$ which is polynomial of degree $\leq n$.*

Proof. Let $F^{(n)}(V)$ denote the subspace of $F(V)$ consisting of those vectors v with the following property:

- $(*)$ There exists a functor $G \in \mathrm{Fun}$ which is polynomial of degree $\leq n$, and a natural transformation $G \to F$ such that v lies in the image of the induced map $G(V) \to F(V)$.

Since the collection of polynomial functors of degree $\leq n$ is stable under sums, we may assume that there exists a single natural transformation $\alpha : G \to F$, where G is polynomial of degree $\leq n$, and the image of each map $G(V) \to F(V)$ coincides with $F^{(n)}(V)$. We can then define $F^{(n)} = \mathrm{Im}(\alpha)$. Then $F^{(n)}$ is a quotient of G, and therefore polynomial of degree $\leq n$. It is easy to see that $F^{(n)}$ has the desired properties. $\square$

Definition 11. A functor $F \in \text{Fun}$ is *analytic* if it is the union of the polynomial subfunctors $\{F^{(n)}\}_{n \geq 0}$. Let Fun^{an} denote the full subcategory of Fun spanned by the analytic functors.

Lemma 12. *The subcategory* $\text{Fun}^{\text{an}} \subseteq \text{Fun}$ *is closed under the formation of quotients, subobjects, and direct sums in* Fun*. In particular,* Fun^{an} *is an abelian category.*

Proof. Suppose given an exact sequence

$$0 \to F' \to F \to F'' \to 0.$$

For each $n \geq 0$, we have an induced exact sequence

$$0 \to F' \cap F^{(n)} \to F^{(n)} \to \text{Im}(F^{(n)} \to F'') \to 0.$$

Since the middle term in this sequence is polynomial of degree $\leq n$, we conclude that the outer terms are also polynomial of degree $\leq n$. Assume that F is analytic. Passing to the direct limit over n, we deduce that F' and F'' can be obtained as the direct limit of sequences of polynomial subfunctors, and are therefore analytic as well.

To prove the assertion regard sums, let us suppose that $F = \oplus_\alpha F_\alpha$. If each F_α can be obtained as the direct limit of a sequence of polynomial subfunctors $F_\alpha^{(n)}$, then F can be obtained as the direct limit of the polynomial functors

$$\oplus_\alpha F_\alpha^{(n)}.$$

□

We will need the following result, whose proof we defer until the next lecture:

Proposition 13. *The category* Fun^{an} *of analytic functors is generated by the objects* $\{\Gamma^n\}_{n \geq 0}$.

Combining this with the results of the previous lecture, we obtain the following:

Corollary 14. *Let* $\mathcal{R} \subseteq \text{Fun}^{\text{an}}$ *denote the full subcategory spanned by the objects* $\{\Gamma^n\}_{n \geq 0}$*. Then we have a pair of adjoint functors*

$$F : \text{Mod}(\mathcal{R}) \to \text{Fun}^{\text{an}}$$

$$G : \text{Fun}^{\text{an}} \to \text{Mod}(\mathcal{R})$$

where F is exact and G is fully faithful.

Proof. The only other point to check is that Fun^{an} is a Grothendieck abelian category. Proposition 13 implies that Fun^{an} has a set of generators, so we just need to know that filtered colimits in Fun^{an} are exact. Since Fun^{an} is stable under colimits in Fun, it suffices to show that filtered colimits in Fun are exact. This follows from the observation that filtered colimits are exact in the category Vect. □

The real point of Corollary 14 is that the category $\text{Mod}(\mathcal{R})$ can be identified with something concrete: namely, the category of unstable $\mathcal{A}$-modules. Let us sketch this identification. According to Proposition 5, we can identify $\mathcal{R}$ with the full subcategory of $\mathcal{U}$ spanned by the modules $\{F(n)\}_{n \geq 0}$. Let M be an $\mathcal{R}$-module: that is, a contravariant functor from $\mathcal{R}$ to the category of abelian groups. We then let M^n denote the value of M on the object $F(n) \in \mathcal{R}$. For every n and every Steenrod operation Sq^I, we have an object $\text{Sq}^I \nu_n \in F(n)$, which we can identify with a map $F(n + \deg(I)) \to F(n)$ in $\mathcal{R}$. This determines a map

$$M^n \to M^{n+\deg(I)}.$$

It is easy to see that this endows M with the structure of a graded $\mathcal{A}$-module. Moreover, since $\text{Sq}^I \nu_n$ vanishes whenever the excess of I is greater than n, we conclude that M is unstable. We leave it to the reader to verify that this determines an equivalence $\text{Mod}(\mathcal{R}) \simeq \mathcal{U}$. We can therefore restate Corollary 14 as follows:

Corollary 15. *There exists a pair of adjoint functors*

$$F : \mathcal{U} \to \mathrm{Fun}^{\mathrm{an}}$$

$$G : \mathrm{Fun}^{\mathrm{an}} \to \mathcal{U}$$

where F is exact and G is fully faithful.

We conclude with an application of Corollary 15. Let V be a finite dimensional $\mathbf{F}_2$-vector space. Let $P_V \in \mathrm{Fun}$ be the functor given by the formula $P_V(W) = \mathbf{F}_2[\mathrm{Hom}(V, W)]$, where $\mathbf{F}_2[\mathrm{Hom}(V, W)]$ denotes the free $\mathbf{F}_2$-vector space generated by the set $\mathrm{Hom}(V, W)$. It follows from Yoneda's lemma that for any $F \in \mathrm{Fun}$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}}(P_V, F) \simeq F(V).$$

The functors P_V form a set of projective generators for Fun. We let I_V denote the dual $DP_{V^\vee}$, so we have isomorphisms

$$\mathrm{Hom}_{\mathrm{Fun}}(F, I_V) \simeq \mathrm{Hom}_{\mathrm{Fun}}(F, DP_{V^\vee}) \simeq \mathrm{Hom}_{\mathrm{Fun}}(P_{V^{vee}}, DF) \simeq DF(V^\vee) = F(V)^\vee.$$

This is evidently an exact functor of F, so that I_V is an injective object of Fun. We observe that I_V can be described by the formula

$$W \mapsto \mathbf{F}_2^{\mathrm{Hom}(W,V)}.$$

Proposition 16. *Let V be a finite dimensional vector space over $\mathbf{F}_2$. Then the functor I_V is analytic.*

Proof. We observe that the category Fun is equipped with a tensor product, described by the formula $(F \otimes F')(V) = F(V) \otimes F'(V)$. If F and F' are polynomial of degrees $\leq n$ and n', respectively, then $F \otimes F'$ is polynomial of degree $\leq n + n'$. It follows that a tensor product of analytic functors is analytic. Moreover, we have a canonical isomorphism $I_{V \oplus V'} \simeq I_V \otimes I_{V'}$. It will therefore suffice to prove Proposition 16 in the case where V has dimension 1. In this case, we can identify I_V with the functor

$$W \mapsto \mathbf{F}_2^{W^\vee}.$$

We now observe that there is a canonical surjection

$$\mathrm{Sym}^* \to I_V,$$

since every function $W^\vee \to \mathbf{F}_2$ is given by some polynomial. Since $\mathrm{Sym}^* \simeq \oplus_n \mathrm{Sym}^n$ is analytic, we conclude that I_V is analytic as desired. □

It follows that for every finite dimensional $\mathbf{F}_2$-vector space V, the functor I_V is an injective object of $\mathrm{Fun}^{\mathrm{an}}$. Since the functor F is exact, we deduce that the functor

$$M \mapsto \mathrm{Hom}_{\mathcal{U}}(M, GI_V) \simeq \mathrm{Hom}_{\mathrm{Fun}}(FM, I_V)$$

is exact. In other words, GI_V is an injective object in the category $\mathcal{U}$ of unstable modules over the Steenrod algebra. It is easy to identify this object: we have

$$(GI_V)^n = \mathrm{Hom}_{\mathrm{Fun}}(\Gamma^n, I_V) \simeq \Gamma^n(V)^\vee = \mathrm{Sym}^n(V^\vee) \simeq \mathrm{H}^n(BV).$$

It is not hard to show that this identification is compatible with the action of the Steenrod algebra. Consequently, we have proven the following:

Proposition 17. *Let V be a finite dimensional vector space over $\mathbf{F}_2$. Then the cohomology ring $\mathrm{H}^*(BV)$ is an injective object of the category $\mathcal{U}$.*