Massachusetts Institute of Technology
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Lecture Notes
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Lecture 1.

1 Manifolds: definitions and examples

Loosely manifolds are topological spaces that look locally like Euclidean space. A little more precisely it is a space *together with* a way of identifying it locally with a Euclidean space which is compatible on overlaps. To formalize this we need the following notions. Let X be a Hausdorff, second countable, topological space.

Definition 1.1. A chart is a pair (U, ϕ) where U is an open set in X and $\phi : U \to \mathbb{R}^n$ is homeomorphism onto it image. The components of $\phi = (x^1, x^2, \dots, x^n)$ are called coordinates.

Given two charts (U_1, ϕ_1) and (U_2, ϕ_2) then we get *overlap or transition* maps

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

and

$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2)$$

Definition 1.2. Two charts (U_1, ϕ_1) and (U_2, ϕ_2) are called compatible if the overlap maps are smooth.

In practice it is useful to consider manifolds with other kinds of regularity. One many consider C^k*-manifolds* where the overlaps are C^k-maps with C^k inverses. If we only require the overlap maps to be homeomorphisms we arrive at the notion of a topological manifold. In some very important work of Sullivan one consider Lipschitz, or Quasi-conformal manifolds.

An *atlas* for X is a (non-redundant) collection $\mathcal{A} = \{(U_\alpha, \phi_\alpha) | \alpha \in A\}$ of pair wise compatible charts . Two atlases are *equivalent* if there their union is an atlas. An atlas $\mathcal{A}$ is called *maximal* if any other atlas compatible with it is contained in it.

Exercise 1. Using Zorn's lemma, show that any atlas is contained in a unique maximal atlas.

Definition 1.3. A smooth n-dimensional manifold is a Hausdorff, second countable, topological space X together with an atlas, $\mathcal{A}$.

1.1 examples

$\mathbb{R}^n$ or any open subset of $\mathbb{R}^n$ is a smooth manifold with an atlas consisting of one chart. The unit sphere

$$S^n = \{(x^0, x^1, \ldots, x^n) | \sum_{i=0}^{n} (x^i)^2 = 1\}$$

has an atlas consisting of two charts $(U_\pm, \phi_\pm)$ where $U_\pm = S^n \backslash \{(\pm 1, 0, 0, \ldots, 0)\}$ and

$$\phi_\pm(x^0, x^1, \ldots, x^n) = \frac{1}{\pm 1 - x_0}(x^1, \ldots, x^n)$$

Real projective space, $\mathbb{RP}^n$, is space of all lines through the origin in $\mathbb{R}^{n+1}$ which we can identify with nonzero vectors up to the action of non-zero scalars so $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{\vec{0}\})/\mathbb{R}^*$. The equivalence class of $(x_0, \ldots, x_n)$ is denoted $[x_0 : x_1 : \ldots : x_n]$. $\mathbb{RP}^n$ has an atlas consisting of $n + 1$ charts. The open sets are

$$U_i = \{[x_0 : x_1 : \ldots : x_n] | x_j \in \mathbb{R}, \text{ and } \mathrm{x_i} \neq 0\}$$

and the corresponding coordinates are

$$\phi_i([x_0 : x_1 : \ldots : x_n]) = (x_1/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i).$$

Similarly we have complex projective space, $\mathbb{CP}^n$, the space of a line through the origin in $\mathbb{C}^{n+1}$. So just as above we have $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{\vec{0}\})/\mathbb{C}^*$. A typical point of $\mathbb{CP}^n$ is written $[z_0 : z_1 : \ldots : z_n]$. $\mathbb{CP}^n$ has a atlas consisting of $n + 1$ charts. The open sets are

$$U_i = \{[z_0 : z_1 : \ldots : z_n] | z_i \neq 0\}$$

and the corresponding coordinates are

$$\phi_i([z_0 : z_1 : \ldots : z_n]) = (z_1/z_i, \ldots, \widehat{z_i/z_i}, z_n/z_i).$$

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Exercise 2. Show that in fact the above construction yield charts.

Notice that in the case of $\mathbb{CP}^n$ the coordinates have values in $\mathbb{C}^n$ and so the overlap maps map an open subset of $\mathbb{C}^n$ to $\mathbb{C}^n$. We can ask that they are holomorphic. We make the following definition.

Definition 1.4. A complex manifold is a Hausdorff second countable topological space X, with an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)|\alpha \in A$ the coordinate functions ϕ_α take values in $\mathbb{C}^n$ and so all the overlap maps are holomorphic.

Let $\mathrm{Gr}_k(\mathbb{R}^n)$ be the space of k-planes through the origin in $\mathbb{R}^n$.

Exercise 3. Show that $\mathrm{Gr}_k(\mathbb{R}^n)$ has an atlas with $\binom{n}{k}$ charts each homeomorphic with $\mathbb{R}^{k(n-k)}$.

Similarly we have $\mathrm{Gr}_k(\mathbb{C}^n)$ the space of all complex k-plane through the origin in $\mathbb{C}^n$.

Exercise 4. Show that $\mathrm{Gr}_k(\mathbb{C}^n)$ has an atlas with $\binom{n}{k}$ charts each homeomorphic with $\mathbb{C}^{k(n-k)}$. Show that we can give $\mathrm{Gr}_k(\mathbb{C}^n)$ the structure of a complex manifold.