Lecture 12.

13 Stratified Spaces

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Definition 13.1. A stratification of a topological space X is a filtraion is a decomposition $X = \bigcup_{i=0}^{n} S_i$ where each of the S_i are smooth manifolds (possibily empty) of dimension i and so that

$$\overline{S_k} \setminus S_k \subset \bigcup_{i=0}^{k-1} S_i.$$

The closure $\overline{S_k}$ is called the stratum of dimension k.

Note that any stratum of a strafied space is a stratified space in its own right.

Stratified spaces are useful because many results about smooth manifolds can be extended to stratified spaces. A good example is the space of matrices $M_{k\times n}$l. The strata are the matrices of rank bounded above by a fixed number. (assume that $k \leq n$)

As an application of this result we will compute the low homotopy groups for the Stiefel manifolds, $\mathrm{St}_k(\mathbb{R}^n)$. Recall that the Stiefel manifold is the space of k-frames in $\mathbb{R}^n$. Given a k-frame $(v_1, v_2, \ldots, v_k)$ we get an injective linear map $A : \mathbb{R}^k \to \mathbb{R}^n$ by sending the standard basis vectors $e_i \to v_i$. In other words we can identify the Stiefel manifold, $V_k(\mathbb{R}^n)$, with the open subset of $\hom(\mathbb{R}^k, \mathbb{R}^n)$ consisting of injective maps. The compliment of $V_k(\mathbb{R}^n)$ has a decomposition according to the dimension of the kernel of the map. To codify this set

$$R_l = \{A \in \hom(\mathbb{R}^k, \mathbb{R}^n) | \mathrm{Rank}(A)) = l\}.$$

We claim that in fact these R_l are submanifolds.

Proposition 13.2. $R_l \subset \hom(\mathbb{R}^k, \mathbb{R}^n)$ *is a smooth submanifold of codimension*

$$(k-l)(n-l).$$

Proof. Fix $A \in S_l$. Write $\mathbb{R}^k = \ker(A) \oplus \mathrm{Ran}(A^*)$ and $\mathbb{R}^n = \ker(A^*) + \mathrm{Ran}(A)$. Then with respect to this decomposition we can write

$$A = \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix}$$

and a nearby matrix as

$$B = A + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

Lemma 13.3. *If $\bar{A} + \alpha$ is invertible then a vector (v, w) is in the kernel of B if and only if $v = -(\bar{A} + \alpha)^{-1}\beta w$ and $(\delta - \gamma(\bar{A} + \alpha)^{-1}\beta)v = 0$*

Proof. If (v, w) is the the kernel of B then

$$(\bar{A} + \alpha)v + \beta w = 0$$

so the first equation is clear. The second equation follows by substituting the first into

$$\gamma v + \delta w = 0$$

□

The lemma implies that the kernel of B is l-dimensional if and only if

$$\delta - \gamma(\bar{A} + \alpha)^{-1}\beta = 0$$

The map

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto \delta - \gamma(\bar{A} + \alpha)^{-1}\beta$$

is clearly a submersion so the preimage of 0, our local model of R_l is a submanifold of codimension

$$\dim(\ker(A)) \dim(\mathrm{Coker}(A)) = (k - l)(n - l).$$

□

We'll use this to do a simple calculation of homotopy groups.

$$\pi_i(\mathrm{St}_k(\mathbb{R}^n) = 0$$

for $i < n - k$. From its definition $\mathrm{St}_k(\mathbb{R}^n)$ can be identified with the space of matrices of maximal rank in $M_{k\times n}$ and so

$$\mathrm{St}_k(\mathbb{R}^n) = M_{k\times n} \setminus (\cup_{l=0}^{k-1} R_l$$

so the problem is to show that a map

$$f : S^i \to \mathrm{St}_k(\mathbb{R}^n)$$

from a sphere of dimension $i < n - k$ is null homotopic. We know that there is a null-homotopy in the larger contractible space of matrices that is to say there is a map

$$h : D^{i+1} \to M_{k\times n}.$$

so that

$$h|_S^i = f.$$

If we can find a homotopy $k : I \times D^{i+1} \to M_{k\times n}$ so that during the homotopy the following two conditions hold.

1. $k|I \times S^i \subset \mathrm{St}_k(\mathbb{R}^n)$
2. $k(\{1\} \times D^{i+1}) \subset \mathrm{St}_k(\mathbb{R}^n)$.

To see that we can do this we will appeal to Sard's theorem. Lets consider the larger family of maps

$$H : M_{k\times n} \times D^{i+1} \to M_{k\times n}$$

given by

$$H(A, x) = A + h(x).$$

If A is small enough then

$$k(t, x) = H(tA, x) = tA + f(x)$$

satisfies the first condition. To see that we can arrange that the second condition is satisfied we note that H is a submersion. Thus the preimages of the R_l's are all submanifolds. Set

$$\tilde{R}_l = H^{-1}(R_l)$$

these are submanifolds of codimension $(k-l)(n-l)$. so they have dimension

$$i+1+nk-(k-l)(n-l)$$

Consider the projection $\tilde{R}_l \to M_{k\times n}$. Provided that for all $l \leq k-1$

$$i+1+nk-(k-l)(n-l) < nk$$

then image of the projection has measure zero. The worst case is $l = k-1$ when the right hand side is

$$i+nk+k-n$$

so that the inequality holds if $i < n-k$. If $(A,x) \ni \tilde{R}_l$ that for all x $f(x) \ni R_l$ completing the proof.