Lecture 14.

15 Whitney's embedding theorem, medium version.

Theorem 15.1. *(Whitney). Let X be a compact n-manifold. Then M admits a embedding in $\mathbb{R}^{2n+1}$.*

Proof. From Theorem [**?**] we can assume that M is embedded in $\mathbb{R}^N$ for some N. To state the next result for a hyperplane $\Pi \subset \mathbb{R}^N$ let $p_\Pi \colon \mathbb{R}^N \to \Pi$ denote the orthogonal projection. Note that the set of hyperplanes in $\mathbb{R}^N$ is a copy of $\mathbb{RP}^{N-1}$ by associating to each hyperplane the orthogonal line. The desired result follows from:

Lemma 15.2. *If $N > 2n+1$ then for a full measure set of hyperplanes $\Pi \subset \mathbb{R}^N$ the composition $p_\Pi \circ \Phi$ is a differentiable embedding of M into Π.*

Proof. Let $\Delta \subset M \times M$ be the diagonal, $\Delta = \{(x,x)|x \in M\}$. Define the map

$$a : M \times M \setminus \Delta \to \mathbb{RP}^{N-1}.$$

which sends distinct points x and x' to the line through the origin parallel to the line passing through x and x' or equivalently the line through 0 and $x - x'$. Notice that $p_\Pi \circ \Phi$ is injective if and only if a misses the line orthogonal to Π. If $2n <$

$N - 1$ then any point in the image of a is a critical value and hence by Sard's theorem the image of has measure zero. Thus the set of then the image of a has measure zero and so the set of hyperplane for which the composition is injective is a Baire set.

Next consider the projectivization of the tangent bundle of M, $\mathbb{P}(TM)$. This is a fiber bundle over M with fiber $\mathbb{RP}^{n-1}$. The total space of the bundle is a smooth manifold of dimension $2n - 1$. Define the map

$$b : \mathbb{P}(TM) \to \mathbb{RP}^{N-1}$$

which sends a line $\ell \in T_x M$ to the line $D_x\Phi(\ell)$ in $\mathbb{R}^N$. Notice that the differential of $p_\Pi \circ \Phi$ is injective precisely when the line orthogonal to Π is not in the image of b. If $2n - 1 < N - 1$ then as above the image of b has full measure.

Thus the set of good planes is the intersection of two sets of full measure and hence had full measure itself. □

□

Notice that the condition on the map b was weaker then the condition on the map a so the proof also proves:

Proposition 15.3. *If M is a closed smooth n-manifold then M immerses into $\mathbb{R}^{2n}$.*

Proof. □

We'll use this theorem to prove the hard version of Whitney's theorem.