16.3 Fredholm Operators

A nice way to think about compact operators is to show that set of compact operators is the closure of the set of finite rank operator in operator norm. In this sense compact operator are similar to the finite dimensional case. One property of finite rank operators that does not generalize to this setting is theorem from linear algebra that if $T\colon X \to Y$ is a linear transformation of finite dimensional vector spaces then

$$\dim(\ker(T)) - \dim(\mathrm{Coker}(T)) = \dim(X) - \dim(Y).$$

Of course if X or Y is infinite dimensional then the right hand side of equality does not make sense however the stability property that the equality implies could be generalized. This brings us to the study of Fredholm operators. It turns out that many of the operators arising naturally in geometry, the Laplacian, the Dirac operator etc give rise to Fredholm operators. The following is mainly from Hörmander

Definition 16.13. Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded linear operator. T is said to be *Fredholm* if the following hold.

1. $\ker(T)$ is finite dimensional.
2. $\mathrm{Ran}(T)$ is closed.
3. $\mathrm{Coker}(T)$ is finite dimensional.

If T is Fredholm define the *index* of T denoted $\mathrm{Ind}(T)$ to be the number $\dim(\ker(T)) - \dim(\mathrm{Coker}(T))$

First let us show that the closed range condition is redundant.

Lemma 16.14. *Let $T : X \to Y$ be a operator so that the range admits a closed complementary subspace. Then the range of T is closed.*

Proof: C be a closed complement for the range. We can assume that T is injective since $\ker(T)$ is a closed subspace and hence $X/\ker(T)$ is a Banach space so we can replace T by the induced map from this quotient. Now consider the map $S : X \oplus C \to Y$ defined by

$$S(x, c) = T(x) + c.$$

S is bounded linear isomorphism and hence by the open mapping theorem S is a topological isomorphism. Thus $\mathrm{Ran}(T) = S(X \oplus \{0\})$ is closed. $\sqcup$.

An important result that will be used over and over again is the openness of invertibility in the operator norm.

Theorem 16.15. *If $T : X \to Y$ is a bounded invertible operator then for all $p\colon X \to Y$ with sufficiently small norm $T + p$ is also invertible.*

Proof. Without loss of generality we can assume $X = Y$ and $T = I$. Then if the norm of p is sufficiently small the Neumann series

$$\sum_{i=1}^{\infty}(-p)^i$$

converges to the inverse of $I + p$. □

We begin with some lemma's

Lemma 16.16. (F. Riesz) *The unit ball B in a Banach space X is compact if and only if B is finite dimensional.*

Proof. See Kerszig Lemma 2.5-4. This is easy for Hilbert spaces but takes a little care for Banach spaces. □

Lemma 16.17. *The following are equivalent:*

1. $\ker(T))$ *is finite dimensional and* $\mathrm{Ran}(T)$ *is closed.*
2. *Every bounded sequence $\{x_i\} \subset X$ with Tx_i convergent has a convergent subsequence.*

Proof: Suppose that 1 holds. Since $\ker(T)$ is finite dimensional it admits a closed compliment C. Since $\mathrm{Ran}(T)$ is closed it is a Banach space so the Banach isomorphism theorem implies $T|_C : C \to \mathrm{Ran}(T)$ is an isomorphism and the result follows. Now suppose that 2 holds. Then a bounded sequence in the kernel has a convergent subsequence so the kernel is finite dimensional. That $\mathrm{Ran}(T)$ is closed follows immediately from 2. □

Let $\mathrm{Fred}(X, Y)$ denote the space of Fredholm operators between X and Y. Also let $\mathrm{Fred}(X)$ be the set of Fredholm operators on X

Lemma 16.18. $\mathrm{Fred}(X, Y)$ *is a open subset of* $\mathcal{B}(X, Y)$ *and the index is a locally constant function on* $\mathrm{Fred}(X, Y)$.

Proof. Let $T : X \to Y$ be a Fredholm operator and let $p : X \to Y$ be an operator with small norm. We can write $X = C + \ker(T)$ and $Y = \mathrm{Ran}(T) + D$. With respect to this decomposition we can write T as a matrix

$$T = \begin{bmatrix} T' & 0 \\ 0 & 0 \end{bmatrix}.$$

and p as the matrix

$$p = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We prove the result by reduction to the finite dimensional situation. In fact we'll prove

Lemma 16.19. *For p sufficiently small there is a linear transformation $A : \ker(T) \to \operatorname{Coker}(T)$ so that*

$$\ker(T + p) \equiv \ker(A) \text{ and } \operatorname{Coker}(T + p) \equiv \operatorname{Coker}(A).$$

In fact the norm of p is small enough then $T + a$ will be invertible and if we set

$$G = \begin{bmatrix} I & -(T' + a)^{-1} b \\ 0 & I \end{bmatrix} \text{ and } H = \begin{bmatrix} I & 0 \\ -c(T' + a)^{-1} & I \end{bmatrix} \tag{7}$$

then

$$H(T + p)G = \begin{bmatrix} T' + a & 0 \\ 0 & -c(T' + a)^{-1} b + d \end{bmatrix}.$$

The lemma follows immediately from this taking $A = -c(T + a)^{-1} b + d$. □

The proof of the lemma proved the following conceptually useful result

Lemma 16.20. *Let $T\colon X \to Y$ be a Fredholm map and $p\colon X \to Y$ a linear map. If p has sufficiently small norm then there are isomorphisms $i\colon X' \oplus K \to X$ and $j\colon Y \to X' \oplus C$ so that*

$$j \circ (T + p) \circ i = \begin{bmatrix} I & 0 \\ 0 & q \end{bmatrix}.$$

for some linear map $q\colon K \to C$.

We'll also need the notion of the adjoint of an operator. If X is a Banach space the dual space of X is the space of all bounded linear functionals on X and is denoted X^*. Given a bounded linear operator $T : X \to Y$ we have get a linear operator

$$T^*\colon Y^* \to X^*$$

by declaring that for $\rho \in Y^*$, $T^*(\rho)$ is the linear functional so which send x to

$$\rho(T(x)).$$

First we give the dual characterization of the norm.

Lemma 16.21. *For all $x \in X$*

$$\|x\| = \sup_{\|\rho\|=1} (|\rho(x)|)$$

Proof. Fix $x_0 \in X$ Certainly $|\rho(x_0)| \leq \|\rho\|\|x_0\|$ so

$$\|x_0\| \geq \sup_{\|\rho\|=1} (|\rho(x_0)|)$$

Define a linear functional $\lambda : \text{span}(x_0) \to \mathbb{R}$ by $\lambda(x_0) = \|x_0\|$ and extending by linearity to the span. Applying the Hahn-Banach theorem to λ and the subadditive function $p(x) = \|x\|$ implies the existence of an extension of λ to the whole of X with

$$|\lambda(x)| \leq \|x\|$$

□

Lemma 16.22. *If T is bounded then T^* is bounded with the same norm*

Proof.

$$\begin{aligned}
\|T\| &= \sup_{x|\|x\|\leq 1} \|Tx\| \\
&= \sup_{x|\|x\|\leq 1} |\sup_{\rho|\|\rho\|\leq 1} \rho(Tx)| \\
&= \sup_{\rho|\|\rho\|\leq 1} \sup_{x|\|x\|\leq 1} |\rho(Tx)| \\
&= \sup_{\rho|\|\rho\|\leq 1} \|T^*(\rho)\| \\
&= \|T^*\|.
\end{aligned}$$

□

We'll need the relationship between the cokernel of T and the kernel of T^*.

Lemma 16.23. *If T has closed range then*

$$\text{Coker}(T)^* \equiv \ker(T^*).$$

Proof. There is a natural map $\ker(T^*) \to \text{Coker}(T)^*$ by sending $\rho \in \ker(T^*)$ to the linear functional $\lambda \in \text{Coker}(T)^*$ where $\lambda(y + TX) = \rho(y)$. This well defined since for all $x \in X$ we have $\rho(Tx) = T^*(\rho)(x) = 0$. Since $\text{Ran}(T)$ is closed, $\text{Coker}(T) = Y/\text{Ran}(T)$ is a Banach space. Given a linear functional $\lambda \in \text{Coker}(T)^*$ so $\lambda\colon Y/\text{Ran}(T) \to \mathbb{R}$ and hence defines a bounded linear functional

$$\rho\colon Y \to Y/\text{Ran}(T) \to \mathbb{R}.$$

Now $(T^*\rho)(x) = \rho(T(x)) = 0$. It is easy to check that this inverts the previous construction. □

Next we observe that compactness is preserved under taking adjoints.

Lemma 16.24. *Let $K\colon X \to Y$ be compact then $K^*\colon Y^* \to X^*$ is compact.*

Proof. This takes a little work. See for example Kreszig *Introductory functional analysis with applications* Theorem 8.2-5. □

Lemma 16.25. *Let $K : X \to X$ be a compact operator. Then $I + K$ is Fredholm.*

Proof: First we coincide the kernel of $I + K$. Let B be the unit ball in $\ker(I + K)$. Then $B = K(B)$ so B is image of a bounded set under a compact operator hence is precompact. But B is closed so B is compact. By Riesz's lemma $\ker(I + K)$ is finite dimensional. Next we show that $\text{Ran}(I + K)$ is closed. By lemma 16.17 it suffices to show that if x_i is a bounded sequence so that $x_i + K_i x_i$ converges to $y \in Y$ then there is $x \in X$ so that $x + Kx = y$. Since $\{x_i\}$ is bounded there is a subsequence x_{i_j} so that $\{Kx_{i_j}\}$ converges. But then $\{x_{i_j}\}$ converges. Thus the operator $I + K$ is a semi-Fredholm. Applying the same arguement to the adjoint $I + K^*$ completes the proof. □

Next we give a useful characterization of Fredholm operators.

Theorem 16.26. *$T : X \to Y$ is Fredholm if and only this a bounded linear operator $R : Y \to X$ so that*

$$RT - I \text{ and } TR - I$$

are compact operators.

Proof. If T is Fredholm then as before we can write

$$X = X' \oplus \ker(T) \text{ and} Y = \text{Ran}(T) \oplus C$$

for closed subspaces $X' \subset X$ and $C \subset Y$. $T|_{X'} : X' \to \text{Ran}(T)$ is an isomorphism so it has and inverse $\tilde{R}$. Extending $\tilde{R}$ to a map $Y \to X$ using the direct sum decomposition gives the required map.

If R exists $\ker(T)$ is finite dimensional from the equation $RT = I + K$. $\text{Ran}(T)$ is finite dimensional from the equation $TR = I + K'$ and the operator is Fredholm. □

Next we consider the composition of Fredholm operators.

Lemma 16.27. *Let $T : X \to Y$ and $S : Y \to Z$ be Fredholm operators. Then $ST : X \to Z$ is Fredholm. Furthermore* $\text{Ind}(ST) = \text{Ind}(T) + \text{Ind}(S)$.

Proof: Since $(ST)^{-1}(0) = T^{-1}(S^{-1}(0))$ we have $\dim(\ker(ST)) \leq \dim(\ker(S)) + \dim(\ker(T))$. Similarly $\dim(\text{Coker}(ST)) \leq \dim(\text{Coker}(S)) + \dim(\text{Coker}(T))$ so the composition is Fredholm.

Next we consider the index assertion. To this end consider the family of operators $A_t : Y \oplus X \to Z \oplus X$ defined by the equation

$$A_t = \begin{bmatrix} cos(t)S & -sin(t)ST \\ sin(t)I & cos(t)T \end{bmatrix}$$

for $0 \leq t \leq 1$. We claim that A_t is a continuous family of Fredholm operators. But

$$A_t = \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} cos(t)I & -sin(t)I \\ sin(t)I & cos(t)I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}.$$

So A_t is the composition of Fredholm operators and hence is Fredholm. Clearly $\text{Ind}(A_0) = \text{Ind}(T) + \text{Ind}(S)$ and $\text{Ind}(A_\pi) = \text{Ind}(ST)$. □