

Lecture 18 and 19

17 Smale's Sard theorem

In the early sixties Smales realized that many of the ideas of differential topology can be applied to aid in the study of PDEs and as part of this program he showed how to generalize Sard's theorem to the infinite dimensional case. First we need to introduce the correct kind of mappings of Banach manifolds.

Definition 17.1. Let X and Y be Banach manifolds and $f: X \rightarrow Y$ a smooth map. We say that f is a Fredholm mapping if for all $x \in X$ the differential

$$d_x f: T_x X \rightarrow T_{f(x)} Y$$

is a Fredholm map

The first problem we run into with trying generalize Sard's theorem is that the notion of measure zero isn't easy to make sense of in an infinite dimensional space however the the complement of a (closed) set of measure zero is an open dense set. The critical set of a map is closed so the image is at worst a countable union of closed sets of measure zero. The complement is a countable intersection of open dense sets. This notion makes sense in an arbitrary topological space. In particular Banach manifold which satisfies the Baire category theorem so such a set is non-empty.

Definition 17.2. Let X be topological space. A set $A \subset X$ is called residual it is a countable intersection of open dense sets.

Thus the Baire category theorem says that a residual subset of a metric space is dense.

Smale's generalization of Sard's theorem is

Theorem 17.3. *Let $f: X \rightarrow Y$ be a smooth mapping of second countable Banach manifolds. Then the set of regular values of f is residual in Y .*

To prove this result we prove a result of independent interest which says that after a change of coordinates a nonlinear Fredholm mapping differs from an linear isomorphism by a nonlinear map between finite dimensional manifolds. We have a kind of analogue of Lemma ??

Lemma 17.4. *Let $f: X \rightarrow Y$ be a Fredholm map. Then for any $x \in X$ there are coordinate charts $\phi: U \subset X \rightarrow B \oplus K \rightarrow$ and $\psi: V \subset Y \rightarrow B \oplus C$ so that*

$$\psi \circ f \circ \phi^{-1}(x, k) = (x, g(x, k)).$$

Proof. This is a local result so we may assume without loss of generality that x is the origin in $\bar{U} \subset X \rightarrow B \oplus K$ and that $f(x)$ is the origin in $\bar{V} \subset Y \rightarrow B \oplus C$ where B is a Banach space, $K = \ker(d_x f)$, and $C = \text{Coker}(d_x f)$. We can also

arrange that $0 \oplus K$ is the kernel of $d_{(0,0)}f$ and that $B \oplus \{0\}$ is complement for the range of $d_{(0,0)}f$ and finally that

$$d_{(0,0)}f = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Write

$$f(x, k) = (a(x, k), b(x, k)).$$

As in the proof of the implicit function theorem consider the map

$$h : U \rightarrow B \oplus K$$

given by

$$h(x, k) = (a(x, k), k).$$

Then the differential of h at $(0, 0)$ is the identity so there is a map q inverting h near the origin. Notice that

$$f \circ q(x, k) = (x, g(x, k))$$

as required. □

Remark 3. This lemma has a very important consequence. Point preimages of Fredholm mappings are locally homeomorphic to the point preimage of a smooth map between finite dimensional manifolds. This the beginning of Kurashi's work in deformation theory for complex manifolds. Kurashi and Smale were contemporaries at Columbia in the early sixties.

We need one more technical lemma.

Definition 17.5. A map $f : X \rightarrow Y$ is said to be locally closed if for all $x \in X$ there is a neighborhood U of x so that $f|_{\bar{U}} : \bar{U} \rightarrow Y$ is a closed map.

Any continuous map from a locally compact space is locally closed. Banach spaces are locally compact if and only if they are finite dimensional.

Lemma 17.6. A Fredholm map $f : X \rightarrow Y$ is locally closed.

Proof. Choose charts as guaranteed by Lemma 17.4 so that we can assume our map has the form

$$f(x, k) = (x, g(x, k))$$

If $A \subset U \subset B \times K$ is closed we must show that $f(A)$ is closed. Let (x_i, c_i) be a sequence in $f(A)$ converging to (x, c) . Then $c_i = g(x_i, y_i)$ for some sequence y_i . Since the y_i are bounded in finite dimensional vector space we can assume that y_i converge. Then clearly (x, c) will be in $f(A)$. \square

We are now ready to prove Smale's Sard theorem.

Proof. Let $f: X \rightarrow Y$ be our Fredholm map. Since X is second countable it is enough to show that there is a covering of X by open sets U so that the regular values of $f|_U$ are residual. In fact we will show that we can find U so that the regular values of $f|_U$ are open and dense. Since f is locally closed and since the critical point set of f is closed there is no problem in choosing U the regular values of $f|_U$ is an open set. Now choose charts about the point in question so that the local representative of f has the form guaranteed by Lemma 17.4. The differential of local representative of f has the form

$$\begin{bmatrix} I & 0 \\ * & d_{(x,k)}g|_K \end{bmatrix}$$

so that $d_{(x,k)}f$ is surjective if and only if $d_{(x,k)}g|_K$ is surjective in other words (x, c) is a regular value for $f|_U$ if and only if c is a regular value of $k \mapsto g(x, k)$ for k in a suitable neighborhood. Thus the intersection of $\mathcal{R}(f|_U)$ with each slice $\{x\} \times C \cap V$ is dense and hence $\mathcal{R}(f|_U)$ is dense. \square