Lecture 2.

2 Smooth maps and the notion of equivalence

Let X and Y be smooth manifolds. A continuous map $f : X \to Y$ is called smooth if for all charts (U, ϕ) for and X and (V, ψ) for Y we have that the composition

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V)$$

is smooth.

Two manifolds X and Y are called *diffeomorphic* if there is a homeomorphism $h : X \to Y$ so that h and h^{-1} are smooth.

3 Standard pathologies.

The condition that X be Hausdorff and second countable does not follow from the existence of an atlas.

The line with two origins. Let X be the quotient space of $\mathbb{R} \times \{0, 1\}$ by the equivalence relation $(t, 1) \equiv (t, 0)$ unless $t = 0$. Then X is not Hausdorff, however X admits an atlas with two charts. Let U_i be the image of $\mathbb{R} \times \{i\}$ in X. These maps invert to give coordinates.

Remark 1. Actually non-Hausdorff spaces which satisfy all the other properties arise in real life for example in the theory of foliations or when taking quotients by non- compact group actions. More work is required to come up with a useful notions to replace that of manifolds in this context.

The long line. Let S_Ω denote the smallest uncountable totally ordered set. Consider the product $X = S_\Omega \times (0, 1]$ with dictionary order topology. Then give X charts as follows. For $(\omega, t) \in X$ if $t \neq 1$ let $U_{(\omega,t)} = \{\omega\} \times (0, 1)$ and $\phi_{(\omega,t)} : U \to \mathbb{R}$ be given by $\phi_{(\omega,t)}(\omega, t) = t$. If $t = 1$ let $S(\omega)$ denote the successor of ω. Set $U_{(\omega,1)} = \{\omega\} \times (0, 1] \sup\{S(\omega)\} \times (0, 1)$ and

$$\phi_{(\omega,t)}(\eta, t) = \begin{cases} t & \text{if } \eta = \omega \\ t + 1 & \text{if } \eta = S(\omega). \end{cases}$$

Exercise 5. Check that overlaps are smooth.

The collection $\{U_{(\omega,1/2)}\}_{\omega \in S_\omega}$ is uncountable and consists of disjoint open sets, so X is not second countable.

Different charts

Consider $\mathbb{R}_1$ denote $\mathbb{R}$ with the following charts $(\mathbb{R}, x)$ and $\mathbb{R}_2$ with the chart $(\mathbb{R}, x^3)$. Identity map $\mathbb{R}_1 \to \mathbb{R}_2$ is smooth but not $\mathbb{R}_2 \to \mathbb{R}_1$. $\mathbb{R}_1$ and $\mathbb{R}_2$ are diffeomorphic by the map $x \mapsto x^3$ thought of as a map from $\mathbb{R}_1 \to \mathbb{R}_2$.

These pathologies are simple problems to keep in mind when thinking about the definitions. There are far more subtle issues that arise. Given a topological manifold we can ask can carry an atlas, and if it carries an atlas how many non-diffeomorphic atlases does it carry. The first observation of this phenomenon is due to John Milnor who showed that the seven-sphere admits an atlas (with two charts!) which is not diffeomorphic to the standard differentiable structure. We'll examine this example later in the course.