Lecture 20.

18 Parametric transversality

An important tool in differential topology is the notion of transversality.

Definition 18.1. $f : M \to N$ is said to be transversal to $Z \subset N$ if for all $m \in M$ we have

$$d_m f(T_m M) + T_{f(m)} Z = T_{f(m)} N.$$

This is sometimes written $f \pitchfork Z$.

Lemma 18.2. *If $f: M \to N$ is transverse to Z then the preimage $f^{-1}(Z)$ is a smooth submanifold of dimension*

$$\dim(M) - \dim(N) + \dim(Z).$$

Proof. Let $x \in f^{-1}(Z)$ and choose charts (U, ϕ) about x and (V, ϕ) about $f(x) \in Z$. We can choose (V, ϕ) so that $\psi(f(x)) = 0$ and $\psi(V \cap Z) \subset \mathbb{R}^z \times \{0\} \subset \mathbb{R}^n$. Let $p \colon \mathbb{R}^n \to \mathbb{R}^{n-z}$ be the projection. Define $g : U \to \mathbb{R}^{n-z}$ by $g(x) = p \circ \psi \circ f|_U(x)$. Then the condition that f is travsversal to Z implies that the origin a regular value of g and hence $g^{-1}(0) = Z \cap U$ is a submanifold. □

Remark 4. Often one can make cleaner statements by introducing the notion of codimension. If $Z \subset N$ is a submanifold we define $\mathrm{codim}(Z) = \dim(N) - \dim(Z)$. It is the number of equations required to cut out Z locally. In the above theorem the codimension of Z and $f^{-1}(Z)$ are the same. (They are each cut out by the same number of equations!)

Our aim is to show that the condition of being transversal is generic in the sense of Sard's theorem. As a model for what we wish to prove consider the following situation.

Let

$$F \colon P \times M \to N$$

be a smooth map.

Theorem 18.3. *Suppose that F is a submersion, i.e. the differential of F is surjective everywhere. Suppose further that P, M and N are finite dimensional. Then for each $p \in P$ we get a map $f_p \colon M \to N$. Given a submanifold Z of N t for a generic $p \in P$ we have f_p is transversal to Z.*

Proof. Since F is a submersion F is transversal to Z so that $S = F^{-1}(Z) \subset P \times M$ is a submanifold. Consider the projection

$$p_1 \colon S \to P.$$

Fix $(p, m) \in S$ and set $n = F(p, m)$ The tangent space of S at (p, m) is $(v, w) \in T_{(p,m)}M$ so that $d_{(p,m)}F(v, w) \in T_n Z$ or equivalently

$$d_m f_p(w) + d_{(p,m)}F(v, 0) \in T_n Z.$$

We claim that p is a regular value of the projection if and only if f_p is transverse to Z. This follows from

Lemma 18.4. *$S = F^{-1}(Z)$ is transverse to $\{p\} \times M$ if and only if f_p is transverse to Z.*

Proof. The first condition is

$$0 \oplus T_m M + (d_{p,m} F)^{-1}(T_n Z) = T_p P \oplus T_m M$$

The second condition is

$$d_{p,m} F(0 \oplus T_m M) + T_n Z = T_n N.$$

Since F is surjective these condition are equivalent.

□

Next we observe that the condition S is transverse to $\{p\} \times M$ is equivalent to the condition that p is regular value of the projection $p_1|_S : S \to P$. The first condition is

$$0 \oplus T_m M + (d_{p,m} F)^{-1}(T_n Z) = T_p P \oplus T_m M$$

while the second is

$$d_{p,m} p_1 : (d_{p,m} F)^{-1}(T_n Z) = T_p P.$$

Since $0 \oplus T_m M$ is the kernel of $d_{p,m} p_1$ is $0 \oplus T_m M$ these conditions are equivalent.

Thus we can appeal to Sard's theorem applied to the projection $p_1 \colon S \to P$ to say that a generic $p \in P$ is a regular value and by the lemma for generic $p \in P$, f_p is transverse to Z. □

Theorem 18.5. *Suppose that F is a submersion, i.e. the differential of F is surjective everywhere. Suppose further that P, M and N are Banach manifolds for each $p \in P$ we get the map $f_p \colon M \to N$ is Fredholm. Given a finite dimensional submanifold Z of N then for a residual set of $p \in P$ we have f_p is transversal to Z.*

Proof. We simply need to check the map $p_1|_S : S \to P$ is Fredholm. To this end we need to inspect the proofs of the two lemmas above. We can sharpen them to the following.

Lemma 18.6. *There an isomorphism*

$$T_p P \oplus T_m M/(0 \oplus T_m M + (d_{p,m} F)^{-1}(T_n Z) \to T_n N/d_{p,m} F(0 \oplus T_m M) + T_n Z$$

Proof. Differential of F induces a map which is easily seen to be an isomorphism using the fact that F is a submersion. □

$$d_{p,m} p_1 : (d_{p,m} F)^{-1}(T_n Z) = T_p P.$$

Lemma 18.7. *There an isomorphism*

$$T_p P \oplus T_m M/(0 \oplus T_m M + (d_{p,m} F)^{-1}(T_n Z) \to T_p P/d_{p,m} p_1 : (d_{p,m} F)^{-1}(T_n Z)$$

Proof. Now the differential of p_1 induces the desired map which is easily seen to be an isomorphism using the fact that p_1 is a submersion. □

These two lemmas tell us that the cokernel of $p_1|_S$ is finite dimensional.

The kernel of the projection $p_1|S$ is the intersection $(0 \oplus T_m M \cap (d_{p,m} F)^{-1}(T_n Z)$. This intersection Fits into a short exact sequence

$$0 \to \ker(d_m f_p) \to (0 \oplus T_m M \cap (d_{p,m} F)^{-1}(T_n Z) \to T_n Z \to 0.$$

and hence is finite dimensional. □

The main application we will have of this result is the following result.

Theorem 18.8. *Let M, N, and Z be smooth manifolds with $Z \subset N$ a submanifold. The set of maps $f : M \to N$ in $C^k(M, N)$ which are transverse to Z is residual in $C^k(M, N)$.*

A little later in the course we will deal with giving $C^k(M, N)$ the structure of a Banach manifold.