

Lecture 29.

21 Canonical forms

21.1 The Lie Derivative

Let M be a vector field on a manifold M . As we say the vector field generates a flow $F_t : M \rightarrow M$ at least locally in M characterized by the condition that for all $x \in M$ we have

$$\frac{d}{dt} F_t(x) = X(F_t(x))$$

Remark 5. Or in words the tangent vector to the curve defined by $t \rightarrow F_t(x)$ at $t = 0$ is $X(x)$.

Using the flow we can differentiate objects on M . For example given a function $f : M \rightarrow \mathbb{R}$ we can compute

$$\frac{d}{dt} f \circ F_t(x).$$

Lemma 21.1. $\frac{d}{dt} f \circ F_t(x)|_{t=0} = D_x f(X(x))$.

Proof. This follows from remark 5. □

We will often write $Xf(x)$ for any of these expressions.

For vector field we can do the same. Here we need to be a little careful about conventions. Suppose that X, Y are vector fields on X . Then we can form their bracket $[X, Y]$. Here it is easiest to think in terms of the action on functions. If f is a C^2 function then we define.

$$[X, Y](f) = X(Yf) - Y(Xf)$$

If in terms of local coordinates $X = a^i \frac{\partial}{\partial x^i}$ and $Y = b^j \frac{\partial}{\partial x^j}$ then

$$[X, Y] = \left(a^i \frac{\partial b^j}{\partial x^i} - b^j \frac{\partial a^i}{\partial x^j} \right) \frac{\partial}{\partial x^j}.$$

The remaining terms dropping out since mixed partials commute.

A little more invariantly in term of a patch we have can represent X and Y by maps

$$X, Y : U \rightarrow B$$

and then

$$X(Yf)(x) = D^2 f(X, Y) + D_x Y(X)f$$

and hence

$$[X, Y]f(x) = (D_x Y(X) - D_x X(Y))f$$

We can also get a path of vector fields at x by considering

$$D_{F_t(x)} F_{-t}(Y(F_t(x)))$$

and we define the *Lie Derivative* to be

$$\mathcal{L}_X Y = \frac{d}{dt} D_{F_t(x)} F_{-t}(Y(F_t(x)))$$

Fortunately we have

Proposition 21.2.

$$\mathcal{L}_X Y = [X, Y]$$

Proof. Let G_s denote the time s flow for the vector field Y . Then

$$D_{F_t(x)} F_{-t}(Y(F_t(x)))f = \frac{d}{ds} f \circ F_{-t} \circ G_s \circ F_t(x)|_{s=0}$$

so that

$$\begin{aligned} \frac{d}{dt} D_{F_t(x)} F_{-t}(Y(F_t(x)))f|_{t=0} &= \frac{d}{dt} \frac{d}{ds} f \circ F_{-t} \circ G_s \circ F_t(x)|_{s=0, t=0} \\ &= \frac{d}{ds} \frac{d}{dt} f \circ F_{-t} \circ G_s \circ F_t(x)|_{s=0, t=0} \\ &= \frac{d}{ds} (-Xf)(G_s(x) + X(f \circ G_s)(x)) \\ &= -Y(Xf) + X(Yf). \end{aligned}$$

As advertised. □

Recall that every square matrix A with complex entries is conjugate to one in Jordan canonical form and many theorems about matrices are obvious once we use this fact. So it is in geometry. We already saw one baby example of canonical forms.

Theorem 21.3. *Let X be a vector field on the Banach manifold M modeled on B . Suppose for some $m \in M$ we have $X(m) \neq 0$. Then there is a chart $\phi : U \times (-\epsilon, \epsilon) \rightarrow M$ about m so that $\phi^*(X) = (0, \frac{d}{dt})$.*

In other words any two vector fields non-zero at a point in M are equivalent in small enough neighborhoods under the action of diffeomorphisms of M fixing the point.