

### Lecture 3.

## 4 The derivative of a map between vector spaces

Let  $f : V \rightarrow W$  be a smooth map between real vector spaces.

**Definition 4.1.** Given  $x \in V$  we say that  $f$  is differentiable at  $x$  if there is a linear map  $L_x : V \rightarrow W$  so that for all  $v \in V$  we have:

$$\|f(x) - f(x') - L_x(x - x')\| = o(\|x - x'\|).$$

Here we using the Landau symbol  $o$  to mean a function  $o : \mathbb{R}_+ \rightarrow \mathbb{R}$  continuous at zero and  $o(0) = 0$ .

Really this is an improper definition. We really need  $V$  and  $W$  to be normed vector spaces and it is natural to require that  $L$  is a continuous linear map. One can try to develop differential calculus on manifolds modelled on general topological vector spaces. A sufficiently general context to work in is that of manifolds modelled on Banach spaces, that is complete normed linear spaces. Essentially of the basic results in differential topology work in this context with the same proofs (as long as proof don't use coordinates)

Notice that map  $L$  in the above definition is unique. If  $L'$  is another such map then

$$\begin{aligned} o(\|x - x'\|) &\geq \|f(x) - f(x') - L(x - x') - (f(x) - f(x') - L'(x - x'))\| \\ &= \|(L - L')(x - x')\| \end{aligned}$$

So  $(L - L')(x - x') = 0$ .

The map  $L$  is called the *differential* of  $f$  at  $x$  and is denoted

$$d_x f \text{ or } D_x f.$$

We say  $f$  is *differentiable* if  $f$  is differentiable at each  $x \in U$  and is *continuously differentiable* if

$$df : U \rightarrow \text{hom}(V, W).$$

is continuous. The second derivative is the derivative of the first derivative and thus is a map

$$d^2 f : U \rightarrow \text{hom}(V, \text{hom}(V, W)).$$

In the finite dimensional case  $\text{hom}(V, \text{hom}(V, W))$  with a subspace of  $\text{hom}(V \otimes V, W)$ . In the infinite dimensional case we need to be more careful but we can identify  $\text{hom}(V, \text{hom}(V, W))$  with bilinear maps from

$$V \rightarrow W.$$

You can read all about this in gory detail in [?]

**Definition 4.2.** A smooth map  $f : X \rightarrow Y$  is called an immersion its differential is everywhere injective. It is called a submersion if its differential is everywhere surjective.

There obvious examples of such maps. Suppose  $m < n$  are positive integers

$$i : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

given by

$$i(x^1, \dots, x_m) = (x^1, \dots, x_m, 0, \dots, 0)$$

is an immersion while

$$s : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

given by

$$s(x^1, \dots, x_m, x_{m+1}, \dots, x_n) = (x^1, \dots, x_m)$$

is a submersion. We will see in the next section that locally these simple examples are completely general.