Lecture 32.

25.2 The Poincaré lemma and homotopy invariance of the DeRham cohomology

There are a bunch of basic forumlas in dealing with forms, the exterior derivative and contraction and the Lie derivative.

Recall that the Lie derivative is defined as follow. Given a vector field v let F_t be its time t flow. By pull back this acts on forms on the manifold. Fixing a point $x \in X$ we can watch what happens to the a form at the point x under the flow, i.e consider the path

$$F_t^*(\omega_{F_t(x)}) \in \Lambda_x^k(X)$$

The derivative at $t = 0$ is called the Lie derivative

$$\mathcal{L}_v \omega = \frac{d}{dt} F_t^*(\omega_{F_t(x)})|_{t=0} \in \Lambda_x^k(X)$$

More generally there is a Lie derivative on tensors. Note that if f is a function then this definition amounts to nothing more that

$$\mathcal{L}_v f = \frac{d}{dt} f \circ F_t(x)_{t=0} = v f(x) = \iota_v df$$

Since the exterior derivative is natural under diffeomorphisms it follows that Lie derivative commutes with d. Hence

$$\mathcal{L}_v df = d\mathcal{L}_v f = d\iota_v df.$$

More generally we have Cartan's formula or the homotopy formula.

$$\mathcal{L}_v \omega = d\iota_v \omega + \iota_v d\omega.$$

We prove this by induction on the degree of the form. We have checked the case of functions. Furthermore it is enough to check that that both sides satisfy the Leibniz rule.

$$\mathcal{L}_v(\omega \wedge \eta) = \mathcal{L}_v(\omega \wedge \eta) = d\iota_v \omega + \iota_v d\omega.$$

Let $i : M \to \mathbb{R} \times M$ be the inclusion $i(x) = (0, x)$ and let $\pi : \mathbb{R} \times M \to M$ be the projection. We claim that the induced maps on cohomology are inverses of each other. Thus we have

Proposition 25.4. *The groups $H^*(M)$ and $H^*(\mathbb{R} \times M)$ are isomorphic.*

To prove this we will construct a map K

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Let $\mathfrak{U} = \{U_\alpha | \alpha \in A\}$be a open cover of a topological space. Using the combinatorics of the cover when can define a complex as follows. Let $C^p(\mathfrak{U})$ be the space of all locally constant functions on $p+1$ fold intersections

$$U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$$

with the symmetry property that if σ is a permutation of $0, \ldots, p$ then

$$f|U_{\alpha_0} \cap \ldots \cap U_{\alpha_p} = \text{sign}(\sigma) f|U_{\alpha_{\sigma(0)}} \cap \ldots \cap U_{\alpha_{\sigma(p)}}.$$

We write $f_{\alpha_0 \ldots \alpha_p}$ for $f|U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$

There is a natural codifferential on such functions

$$\delta : C^p(\mathfrak{U}) \to C^{p+1}(\mathfrak{U})$$

defined by the formula

$$(\delta f)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i f_{\alpha_0 \ldots \hat{\alpha}_i \alpha_{p+1}}|_{U_{\alpha_0} \cap \ldots \cap U_{\alpha_{p+1}}}$$

If we order A then we can consider only ordered intersections and define a similary complex which has isomorphic cohomology. In practice this is how one work but the first definition is choice free so a bit prefereable.

Example. Think of S^2 as the boundary of tetrahedron. Cover S^2 by the four open which are the complements of the four closed two dimensional faces. If we label these sets U_1, U_2, U_3, U_4 then the non empty two fold intersections are

$$U_1 \cap U_2, U_1 \cap U_3, U_1, \cap U_4, U_2 \cap U_3, U_2 \cap U_4, U_3 \cap U_4.$$

and the non-empty three fold intersections are

$$U_1 \cap U_2 \cap U_3, U_1 \cap U_2 \cap U_4, U_1 \cap U_3 \cap U_4, U_2 \cap U_3 \cap U_4$$

the four-fold intersection is empty.

Then all interections are connected and the complex is

$$\mathbb{R}^4 \mapsto \mathbb{R}^6 \mapsto \mathbb{R}^4$$

with the maps

$$\delta_0(f_1, f_2, f_3, f_4) = (f_1 - f_2, f_1 - f_3, f_1 - f_4, f_2 - f_3, f_2 - f_4, f_3 - f_4) \quad (8)$$

and

$$\delta_1(f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}) = (f_{23} - f_{13} + f_{12}, f_{24} - f_{14} + f_{12}, f_{34} - f_{14} + f_{13}, f_{34} - f_{24} + f_{23}) \quad (9)$$

The kernel of δ_0 is clearly the constant functions. Cokernel of δ_1 is one dimensional and hence we have $\check{H}^*(\mathfrak{U}) = \mathbb{R}, 0, \mathbb{R}$.