26.1 refinement

By a refinement $\mathfrak{V}$ of an open cover $\mathfrak{U}$ we mean a $\mathfrak{V} = \{V_\beta | \beta \in B\}$ and a map $r : B \to A$ so that for all $\beta \in B$ we have $V_\beta \subset U_{r(\beta)}$. If we have a refinement then there is a chain map of the Čeck complexes.

$$\tilde{r} : \check{C}^p(\mathfrak{U}) \to \check{C}^p(\mathfrak{V})$$

given by the formula

$$\tilde{r}(\{f_{\beta_0\beta_1\ldots\beta_p}\}) = \{f_{\beta_{r(0)}\beta_{r(1)}\ldots\beta_{r(p)}}|_{V_{\beta_{r(0)}\beta_{r(1)}\cdots\beta_{r(p)}}}\}$$

Thus there is a map

$$\tilde{r}^* : \check{H}^*(\mathfrak{U}) \to \check{H}^*(\mathfrak{V}).$$

Thus we have an directed system (well really need to check that if we have two refinements $\mathfrak{V}, r$ and $\mathfrak{V}, r'$ then the induced maps $\tilde{r}$ and $\tilde{r}'$ are the same.) The direct limit of this system is called the Čech cohomology of X.

27 The acyclicity of the sheaf of p-forms.

Then we can consider another version of the of the Čech complex. That is we define $\check{C}^p(\mathfrak{U}, \Omega^q)$ to be all colletions of q-forms $\omega_{\alpha_0\ldots\alpha_p}$ defined on $U_{\alpha_0\ldots\alpha_p}$ with the symmetry properties above. The same formula above defines a differential mapping

$$\check{C}^p(\mathfrak{U}, \Omega^q) \to \check{C}^{p+1}(\mathfrak{U}, \Omega^q)$$

Given an open cover $\mathfrak{U}$ consider the Čech complex

$$\ldots \check{C}^{k-1}(\mathfrak{U}; \Omega^p) \xrightarrow{\delta} \check{C}^k(\mathfrak{U}; \Omega^p) \xrightarrow{\delta} \check{C}^{k+1}(\mathfrak{U}; \Omega^p) \xrightarrow{\delta}$$

Lemma 27.1. *This sequence is exact so long as* $k > 0$*.*

Proof. Fix a partition of unity $\{\phi_\beta | \beta \in B\}$ subordinate to $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$. The supports of the ϕ_β are a refinement of the U_α and we choose a refinement function $r : B \to A$ so that $supp(\phi_\beta) \subset r(\beta)$. Define

$$K : \check{C}^{k+1}(\mathfrak{U}; \mathcal{S}_{\Omega^p}) \to \check{C}^k(\mathfrak{U}; \mathcal{S}_{\Omega^p})$$

by

$$K(\omega)|_{U_{\alpha_0\alpha_1\ldots\alpha_{k-1}}} = \sum_{\beta \in B} \phi_\beta \omega|_{U_{r(\beta)\alpha_0\alpha_1\ldots\alpha_{k-1}}}$$

Since the supports of the ϕ_βs are locally finite by definition of partition of unity this is well defined. Now consider where $k \geq 1$

$$
\begin{aligned}
(\delta K + K\delta)\omega|_{U_{\alpha_0\alpha_1\ldots\alpha_k}} &= \sum_{i=0}^{k}(-1)^i K(\omega)|_{U_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_k}} + \sum_{\beta\in B}\phi_\beta(\delta\omega)|_{U_{r(\beta)\alpha_0\alpha_1\ldots\alpha_k}} \\
&= \sum_{i=0}^{k}(-1)^i \sum_{\beta\in B}\phi_\beta\omega|_{U_{r(\beta)\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_k}} \\
&+ \sum_{\beta\in B}\phi_\beta\omega|_{U_{\alpha_0\alpha_1\ldots\alpha_k}} - \sum_{\beta\in B}\sum_{j=0}^{k}(-1)^j\phi_\beta\omega|_{U_{r(\beta)\alpha_0\ldots\hat{\alpha}_j\ldots\alpha_k}}\Big) \\
&= \omega|_{U_{\alpha_0\alpha_1\ldots\alpha_k}}.
\end{aligned}
$$

We have used that the sum if locally finite to rearrange the order summation. Thus we have proved the identity is cochain homotopic to zero and so the cohomology groups are zero. Note that if $k = 0$ then we simple get zero and the arguement proves nothing. □

Definition 27.2. A sheaf that admits partitions of unity is called fine.