28 The Poincaré Lemma implies the equality of Čech cohomology and de Rham cohomology

The proof here is modelled on the presentation of Weil's proof (see Weil, Andr "Sur les thormes de de Rham." Comment. Math. Helv. 26, (1952). 119–145.) in Principles of Algerbraic Geometry by Griffiths and Harris published by John Wiley and Sons, Inc.

The scheme of the proof is to first restrict attention to countable good covers which we assume to be cofinal in the set of countable covers.

The Poincaré lemma tells us that that for a contractible open set U

$$\mathbb{R} \hookrightarrow \Omega^0(U) \stackrel{d}{\to} \Omega^1(U) \stackrel{d}{\to} \Omega^2(U) \stackrel{d}{\to} \dots$$

is a long exact sequence. We introduce the notation $\mathcal{Z}^p$ for the closed p-forms so that

$$\mathcal{Z}^p(U) = \{\theta \in \Omega^p(U) | d\theta = 0\}$$

then we can break up this long exact sequence into short exact sequences.

$$0 \to \mathcal{Z}^p(U) \hookrightarrow \Omega^p(U) \stackrel{d}{\to} \mathcal{Z}^{p+1}(U) \to 0.$$

Note that $\mathcal{Z}^0(U)$ is the constant function so a copy of $\mathbb{R}$. These induce long exact sequences in cohomology.

$$\check{H}^{i-1}(M; \Omega^p) \to \check{H}^{i-1}(M; \mathcal{Z}^{p+1}) \to \check{H}^i(M; \mathcal{Z}^p) \to \check{H}^i(M; \Omega^p) \to$$

We have seen that $\check{H}^i(M; \Omega^p) = \{0\}$ for $i > 0$ and hence

$$\check{H}^i(M; \mathcal{Z}^p) \equiv \check{H}^{i-1}(M; \mathcal{Z}^{p+1})$$

for $i \geq 2$. Now by definition we the p-th Čech cohomology group of M is

$$\check{H}^p(M; \mathbb{R}) = H^p(M; \mathcal{Z}^0).$$

Repeated applying the isomorphism above we have

$$\check{H}^p(M; \mathbb{R}) \approx H^1(M; \mathcal{Z}^{p-1}).$$

Now consider the beginning of the long exact sequence

$$\begin{array}{c} \to H^1(M; \mathcal{Z}^{p-1}) \to H^1(M; \Omega^p) \\ 0 \to H^0(M; \mathcal{Z}^{p-1}) \to H^0(M; \Omega^{p-1}) \to H^0(M; \mathcal{Z}^p) \end{array}$$

which becomes

$$0 \to \mathcal{Z}^{p-1}(M) \to \Omega^{p-1}(M) \stackrel{d}{\to} \mathcal{Z}^p(M) \to H^1(M; \mathcal{Z}^{p-1}) \to 0$$

Thus

$$H^1(M; \mathcal{Z}^{p-1}) \approx \mathcal{Z}^p(M)/d\Omega^{p-1}(M) = H^p_{deR}(M; \mathbb{R}).$$

Thus we have proved that there is a natural isomorphism

$$\check{H}^p(M; \mathbb{R}) \approx H^p_{deR}(M; \mathbb{R}).$$