Lecture 35.

29 The immersion theorem of Smale

Let $\mathbf{Imm}(X, Y)$ denote the space of immersion of X into Y. Fixing base points $x \in X$ and $y \in Y$ and an injection $\xi : T_x X \to T_y Y$. let $\mathbf{Imm}_*(X, Y)$ be the space of base point preserving immersions in the sense that

$$f(x) = y, \qquad d_x f = \xi.$$

Let $\mathbf{Imm}^1(X, Y)$ denote the space of pair (f, f') where $f : X \to Y$ is an immersion and f' is a section of $f^*(TY) \to X$ with the property that $f^{'}(x) \ni \mathrm{Ran}(d_x f)$ and let $\mathbf{Imm}^1_*(X, Y)$ denote the based version. Here is the proof of the covering homotopy property of the natural map

$$\pi : \mathbf{Imm}(D^k, \mathbb{R}^n) \to \mathbf{Imm}^1(S^{k-1}, \mathbb{R}^n)$$

where $\pi(f) = (f|_{S^{k-1}}, \frac{\partial f}{\partial n}|_{S^{k-1}})$.

The idea of the proof is the following. The condition of being an immersion is open and there is certainly a section of π (indeed linear) if we disregard the immersion condition so we can alway lift a given a homotopy for a short time where the time depends on how close to failing to be an immersion the time zero lift is and on how big the derviatives of the section are. Smale's trick is morally to essentially homotope the time zero lift to be very much inside the space of immersion. Then he can lift the homotopy a fixed amount along the time parameter in the homotopy See "The classification of immersions of spheres in Euclidean Spaces" by Stephen Smale in the Annals of Mathematics Vol. 69, No. 2, March 1959, pg 327.