Lecture 5.

6 More examples.

The orthogonal group. Let

$$O(n) = \{A \in \mathrm{M}_{n\times n}(\mathbb{R}) | AA^T = I\}.$$

be the group of orthogonal transformations of $\mathbb{R}^n$. We claim that the orthogonal group is a smooth manifold. To see this consider the map

$$f : \mathrm{M}_{n\times n}(\mathbb{R}) \to \mathrm{S}ym_n(\mathbb{R})$$

given by

$$f(A) = AA^T$$

where $\mathrm{S}ym_n(\mathbb{R})$ denotes the space of symmetric $n \times n$ matrices. Then $O(n) = f^{-1}(I)$ so it suffices to show that I is a regular value. The differential of f is

$$D_A f(B) = AB^T + BA^T.$$

and we must show that it is surjective. Fix $A \in O(n)$ and choose $C \in Sym_n(\mathbb{R})$. If we take $B = \frac{1}{2}CA$ then

$$D_A f(B) = \frac{1}{2}(AA^T C^T + CAA^T) = C$$

as required.

Let prove existence and uniqueness theorem for ODEs using the inverse function theorem. Let $X : B \to B$ be a smooth map of Banach spaces. We would like so see that the differential equation

$$\frac{dx}{dt} = X(x)x(0) = x_0$$

has a unique solution for all $x_0 \in B$. Define a map

$$F : C^1([0,\epsilon], B) \to C^0([0,\epsilon], B) \times B$$

by

$$F(x) = (\frac{dx}{dt} - X(x), x(0))$$

Lemma 6.1. *If X is K-Lipschitz so is $F : C^0 \to C^0$. If X is C^1 with uniformly bounded*

Proof. $|X(x) - X(x')|_{C^0} \leq K|x - x'|_{C^0}$ if X is K-Lipschitz. We also have that

$$|X(x) - X(x') - D_x X(x - x')| \leq o_x(x - x')$$

□