Lecture 6.

7 Vector bundles and the differential

Consider the Grassman manifold say $\mathrm{Gr}_2(\mathbb{R}^4)$ of two planes in $\mathbb{R}^4$. Let

$$\gamma = \{(\Pi, x) \in \mathrm{Gr}_2(\mathbb{R}^4) \times \mathbb{R}^4 | x \in \Pi\}.$$

Let $p\colon \gamma \to \mathrm{Gr}_2(\mathbb{R}^4)$ be the natural projection. The fibers of p, $p^{-1}(\Pi)$ are vector spaces (in this case over the reals).

This is an example of a vector bundle. We'll give the definition appropriate for the world of smooth manifolds. There is an obvious version of the definition for more general topological spaces.

Definition 7.1. Let V be a vector space (over the reals, complexes or quaternions.) A vector bundle with fiber V is a triple (E, B, p) where E and B are smooth manifolds and $\pi : E \to B$ is a smooth map. For each $b \in B$, $p^{-1}(b)$ has the structure of a vector space over the same field as V and for each $b \in B$ there is an open set U and a smooth map $\phi\colon p^{-1}(U) \to V$ which is linear isomorphism on each fiber. In addition the map $\tau_\phi : p^{-1}(U) \to U \times V$ given by $\tau_\phi(e) = (p(e), \phi(e))$ is a diffeomorphism.

The map τ_ϕ is called a *local trivialization*.

Example 7.2. Let

$$\gamma = \{(\Pi, v) \subset \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n | v \in \Pi\}.$$

We claim as the natural projection $p : \gamma \to \mathrm{Gr}_k(\mathbb{R}^n)$ has the structure of a vector bundle with fiber $\mathbb{R}^k$. Let $\phi : U_\Pi \to \hom(\Pi, \Pi^\perp)$ be one of our charts. Then ϕ^{-1} is given by $A \to \Gamma_A \subset \mathbb{R}^n = \Pi \oplus \Pi^\perp$ where Γ_A denotes the graph of A. The map $\phi : p^{-1}(U_\Pi) \to \Pi$ is simply the orthogonal projection.

A very important notion is the transition function. Suppose we are given two trivializations $\tau_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times V$ and $\tau_\beta\colon p^{-1}(U_\beta) \to U_\beta \times V$. Then get a map

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Gl}(V).$$

defined as follows. If

$$\tau_\alpha(v) = (p(v), \phi_\alpha(v)) \text{ and } \tau_\beta(v) = (p(v), \phi_\beta(v))$$

then

$$g_{\alpha\beta}(p(v))\phi_\beta(v) = \phi_\alpha(v).$$

The transition function satisfy the *cocycle condition*: If we have three trivializations $\tau_\alpha, \tau_\beta, \tau_\gamma$ over open sets $U_\alpha, U_\beta, U_\gamma$ then for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$$

A vector bundle is determined its transition functions and give an open cover $\{U_\alpha\}$ and a collection of functions

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Gl}(V).$$

satisfying the cocycle condition we can construct a vector bundle.

7.1 New vector bundles from old

We can get new vector bundles from old bundles in a number of ways. Given $p_1 \colon V_1 \to X$ and $p_2 \colon V_2 \to X$ we can take direct (or Whitney) sum to get a bundle $V_1 \oplus V_2 \to X$ whose fiber above x is $p_1^{-1}(x) \oplus p_2^{-1}(x)$. Another important operation is the pullback. Suppose we have $p \colon V \to X$ and $f : Y \to X$ a smooth map. Then we can form a vector bundle over Y as follows. The total space denoted $f^*(V)$ is:

$$f^*(V) = \{(y, v) \,|\, f(y) = p(v)\}$$

and projection

$$f^*(p)(y, v) = y.$$