Lecture 7.

7.2 The tangent bundle

Let M be a smooth manifold. We will associate to M a bundle TM. We will do this concretely but there are many ways of doing this. You should read about them all!!!

We know what a tangent vector in $\mathbb{R}^n$.

Definition 7.3. A tangent vector to M at x is the equivalence class of all pairs $v, (U, \phi)$ where (U, ϕ) is a chart about x and v is a tangent vector to $\mathbb{R}^n$ at $\phi(x)$. We say that $v', (U', \phi')$ is equivalent to $v, (U, \phi)$ if

$$v' = d_{\phi(x)}(\phi' \circ \phi^{-1})(v).$$

The tangent bundleTM to M is the set of all tangent vectors.

In other words the tangent bundle to M is bundle determined by choosing an atlas $\{(U_\alpha, \phi_\alpha) | \alpha \in X\}$ and taking as transition functions

$$g_{\alpha\beta}(x) = d_{\phi_\beta(x)}(\phi_\alpha \circ \phi_\beta^{-1})(v).$$

Given a chart (U, ϕ) we get coordinates $x^1, x^2, \ldots, x^n$ on U. A typical tangent X vector is written as

$$X = a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + \ldots a^n \frac{\partial}{\partial x^n}.$$

reminding us that we can differentiate function using tangent vectors. Given $f : M \to \mathbb{R}$ and a tangent vector at $x\ inM$ we define

$$Xf(x) = a^1 \frac{\partial f \circ \phi^{-1}}{\partial x^1}(\phi(x)) + a^2 \frac{\partial f \circ \phi^{-1}}{\partial x^2}(\phi(x)) + \ldots + a^n \frac{\partial f \circ \phi^{-1}}{\partial x^n}(\phi(x)). \tag{2}$$

in other word the usual directional derivative of $f \circ \phi^{-1}$.

Given a smooth map $f : M \to N$ we can define the differential of f as a map

$$Df : TM \to TN.$$

Given x in M and $X = (v, (U, \phi))$ a tangent vector and a chart (V, ψ) about $f(x)$ set $D_x f(X)$ to be the equivalence class of the vector

$$D_{\phi(x)}(\psi \circ f \circ \phi^{-1})(v)$$

and the chart, (V, ψ) or in terms of coordinates if we write

$$\psi \circ f \circ \phi^{-1}(x^1, x^2, \ldots, x^n) = (f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n))$$

then the matrix of Df is

$$[\frac{\partial f^i}{\partial x^j}].$$