Lecture 8.

8 Connections

We motivate the introduction of connections in a vector bundle as a generalization of the usual directional derivative of functions on a manifold. Given a vector field X and a function f on a manifold M, its directional derivative is a new function as in equation (2). Thus we have a map

$$C^\infty(M; TM) \times C^\infty(M) \to C^\infty(M).$$

This map has the following properties.

$$X(fg) = fXg + gXf \tag{3}$$
$$(\alpha X + \beta Y)f = \alpha Xf + \beta Yf \tag{4}$$

where X and Y are smooth vector fields and α, β, f and g are smooth functions.

If we try to generalize this to a directional derivative on sections of a vector bundle we would like a map

$$C^\infty(M; TM) \times C^\infty(M; E) \to C^\infty(M; E).$$

This map is using denoted

$$(X, s) \mapsto \nabla_X s$$

We can no longer multiply sections of a vector bundle but we can multiply sections of a vector bundle by functions. The appropriate generalization of the two rules about are

$$\nabla_X fs = f\nabla_X s + (Xf)s \tag{5}$$
$$\nabla_{\alpha X + \beta Y} s = \alpha \nabla_X s + \beta \nabla_Y f \tag{6}$$

9 Partitions of unity

Given an open cover, $\{U_\alpha | \alpha \in A\}$ of a topological space X we say that a collection of function $\beta_\alpha \colon X \to \mathbb{R}_{\geq 0}$ is a *partition of unity* if

1. For all $\alpha \in A$ Support$(\beta_\alpha) \subset U_\alpha$
2. The collection $\{\text{Support}(\beta_\alpha) | \alpha \in A\}$ is locally finite, that is to say for all $x \in X$ there is a neighborhood of x meeting only finitely many of members of the collection.
3. For all $x \in X$ we have
$$\sum_{\alpha \in A} \beta_\alpha(x) = 1.$$

Smooth manifolds have smooth partitions of unity.

10 The Grassmanian is universal

We say that bundle is of *finite type* if there is a finite set of trivializations whose open sets cover. In this section we will prove the following theorem.

Theorem 10.1. *Let $E \to M$ be a vector bundle of finite type. Then for some N large enough there is a map*

$$f : M \to \text{Gr}_k(\mathbb{R}^N).$$

Proof. Let $\{(U_i, \tau_i) | i = 1, \ldots m\}$ be a collection of trivializations so that the U_i cover. Write the trivializations as $\tau_i(e) = (p(e), \phi_i(e))$ as before. Choose a partition of unity $\{\beta_i | i = 1, \ldots, m\}$ subordinate to the U_i. Then define

$$\Phi : E \to \mathbb{R}^{mk}$$

by the formula

$$\Phi(e) = (\beta_1(p(e))\phi_1(e), \beta_2(p(e))\phi_2(e), \ldots, \beta_m(p(e))\phi_m(e)).$$

Φ is well defined by the support condition on the partition of unity. Φ is linear on each fiber of E as the ϕ_i are. Φ is injective on each fiber since for each $b \in B$

there is a β_i with $\beta_i(b) \neq 0$. Thus for each point $b \in B$ we have that $\Phi^{-1}(p^{-1}(b))$ is a k-plane in $\mathbb{R}^{mk}$. So we can now define

$$f : B \to \mathrm{Gr}_k(\mathbb{R}^{mk})$$

by

$$f(b) = \Phi(p^{-1}(b)).$$

Exercise 6. Check that this map is smooth. In other words write the map down in charts on the domain and range.

We claim that $f^*(\gamma_k)$ is isomorphic to E. Consider the map

$$\tilde{\Phi} : E \to B \times \gamma_k$$

given by

$$\tilde{\Phi}(e) = (p(e), (\Phi(p^{-1}(p(e))), \Phi(e))).$$

From the definition of f this maps E to $f^*(\gamma_k)$.

Exercise 7. Check that this is an isomorphism.

$\square$