18.969 Topics in Geometry, MIT Fall term, 2006

Problem sheet 1

Exercise 1. Let m_f be the operation of multiplication by the function $f \in C^\infty(M)$. Since $[d, m_f] = e_{df}$, where $e_{df} : \rho \mapsto df \wedge \rho$, this means that the symbol sequence associated to the de Rham complex

$$\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)$$

is the "Koszul complex" of wedging by a 1-form ξ:

$$\wedge^{k-1}T^* \xrightarrow{e_\xi} \wedge^k T^* \xrightarrow{e_\xi} \wedge^{k+1}T^*$$

Show that for $\xi \neq 0$ the above is an exact sequence, i.e. $\ker e_\xi = \mathrm{im}\, e_\xi$.

For the significance of this, see section 3, Atiyah and Bott: "A Lefschetz Fixed Point Formula for Elliptic Complexes: II. Applications", Annals of Mathematics, 2nd ser., Vol. 88, No. 3 (1968) pp. 451-491. Available on JSTOR.

Exercise 2. Let $X, Y \in C^\infty(T)$ and $\pi \in C^\infty(\wedge^2 T)$, so that, in a coordinate patch with coordinates x_i, we have $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$ and $\pi = \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. Compute $[X, Y]$, $[\pi, X]$, and $[\pi, \pi]$ in coordinates.

Exercise 3 (*)**.** We saw that vector fields $X \in C^\infty(T)$ determine degree -1 derivations of the graded commutative algebra of differential forms, i.e.

$$i_X \in \mathrm{Der}^{-1}(\Omega^\bullet(M)).$$

Also, the exterior derivative is a derivation of degree $+1$:

$$d \in \mathrm{Der}^1(\Omega^\bullet(M)).$$

As a result, the graded commutator $L_X = [i_X, d]$, called the *Lie derivative*, is also a derivation:

$$L_X \in \mathrm{Der}^0(\Omega^\bullet(M)).$$

Are there any more derivations? Describe the entire graded Lie algebra of derivations completely.

Useful reference: Michor, "Remarks on the Frölicher-Nijenhuis bracket", Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl. 16, 1987. Available on

`http://www.mat.univie.ac.at/~michor/listpubl.html`

Exercise 4. Let $\omega \in C^\infty(\wedge^2 T^*)$ be nondegenerate, so that the map $\omega : T \longrightarrow T^*$ defined by

$$\omega : X \mapsto i_X \omega$$

is invertible. Show that this is only possible if $\dim T = 2n$ for some integer n.

Then $\det \omega : \det T \longrightarrow \det T^*$, or in other words

$$\det \omega \in \det T^* \otimes \det T^*.$$

Show that $\det \omega = (\text{Pf } \omega)^2$, where

$$\text{Pf } \omega = \tfrac{1}{n!}\omega^n.$$

Exercise 5. Show that S^4 has no symplectic structure. Show that $S^2 \times S^4$ has no symplectic structure.

Exercise 6 (*)**.** Let $P \in C^\infty(\wedge^2 T)$ and let $\xi_1, \xi_2, \xi_3 \in \Omega^1(M)$.

- Show that

$$i_P(\xi_1 \wedge \xi_2 \wedge \xi_3) = i_P(\xi_1 \wedge \xi_2)\xi_3 + i_P(\xi_2 \wedge \xi_3)\xi_1 + i_P(\xi_3 \wedge \xi_1)\xi_2.$$

- Defining the bracket on functions $\{f, g\} = i_P(df \wedge dg)$, show that $\{\cdot, \cdot\}$ satisfies the Jacobi identity if and only if $[P, P] = 0$.
- Let $\omega \in C^\infty(\wedge^2 T^*)$ be nondegenerate. Then prove that $d\omega = 0$ if and only if $[\omega^{-1}, \omega^{-1}] = 0$, where $\omega^{-1} \in C^\infty(\wedge^2 T)$ is obtained by inverting ω as a map $\omega : T \longrightarrow T^*$.

Exercise 7. Write the Poisson bracket $\{f, g\}$ in coordinates for $\pi = \pi^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$.

Exercise 8 (*)**.** Let $v \in C^\infty(\wedge^2 T^*)$ be the standard volume form of the outward-oriented S^2, and let $h \in C^\infty(S^2)$ be the standard height function taking value 0 along the equator and ± 1 on the poles. Define $\pi = hv^{-1}$ and show π is a Poisson structure. Determine $d\pi$ as a section of $T^* \otimes \wedge^2 T = T$ along the vanishing set of π and draw a picture of Hamiltonian flow by the function h.

Exercise 9. Describe Hamiltonian flow in the symplectic manifold T^*M by the Hamiltonian $H = \pi^* f$, where $\pi : T^*M \longrightarrow M$ is the natural projection and $f \in C^\infty(M)$. Also, show that a coordinate chart $U \subset M$ determines a system of n independent, commuting Hamiltonians on $T^*U \subset T^*M$.

Exercise 10 (*)**.** State the Poincaré lemma for the de Rham complex, thought of as a complex of sheaves. State the Poincaré lemma (sometimes called the Dolbeault lemma) for the Dolbeault complex $(\Omega^{p,\bullet}(M), \overline{\partial})$. Explain why these lemmas imply that the cohomology with values in the sheaf of locally constant functions and holomorphic p-forms can be computed by

$$H^q(M, \mathbb{R}) = \frac{\ker d|_{\Omega^q}}{\text{im} d|_{\Omega^{q-1}}},$$

$$H^q(M, \Omega^p_{hol}) = \frac{\ker \overline{\partial}|_{\Omega^{p,q}}}{\text{im}\overline{\partial}|_{\Omega^{p,q-1}}},$$

Finally, determine if the Poisson cohomology complex $(C^\infty(\wedge^p T), d_\pi)$ satsifies, in general, the Poincaré lemma.