18.969 Topics in Geometry, MIT Fall term, 2006

Problem sheet 3

Exercise 1. Let $g_{ij} : U_i \cap U_j \longrightarrow S^1$ be the transition functions of a S^1 principal bundle P or equivalently a unitary complex line bundle. Show that the choice of a connection on this bundle is equivalent to finding 1-forms $A_i \in \Omega^1(U_i)$ such that, on double overlaps,

$$\delta A = -id \log g,$$

where $\delta A = A_i - A_j$ and $d \log g = g_{ij}^{-1} dg_{ij}$. Show also that dA_i define a global closed 2-form F equal to the curvature of the connection. When is F exact?

Show, using a partition of unity, that such a connection may always be chosen.

Recall that the transition functions g_{ij} may be used to construct a Lie algebroid of the form

$$0 \longrightarrow 1 \longrightarrow A \longrightarrow T \longrightarrow 0 \ ,$$

where 1 denotes the trivial real line bundle. Show explicitly that A is isomorphic to the Atiyah algebroid TP/S^1 as defined in the lecture. Furthermore, show that the choice of connection is equivalent to choosing a splitting $s : T \longrightarrow A$ of this sequence. Finally, show that

$$s^*[s(X), s(Y)] = F(X, Y),$$

where $s^* : A \longrightarrow 1$ is the induced projection and F is the curvature of the connection.

Exercise 2. Let (L_{ij}, θ_{ijk}) define a S^1-gerbe in a covering (U_i). A 0-connection for this gerbe consists of a choice of connections ∇_{ij} on L_{ij} such that $\theta_{ijk} \in L_{ij} \otimes L_{jk} \otimes L_{ki}$ is flat in the induced connection on the triple tensor product.

A 1-connection is then a choice of real 2-forms B_i satisfying

$$\delta B = F$$

on double overlaps, where $\delta B = B_i - B_j$ and F is the curvature of ∇_{ij}. Then the 3-form $H = dB_i$ is globally defined; it is the curvature of the gerbe connection.

Show that 0-connections and 1-connections may always be chosen on any S^1-gerbe. Hint: use partitions of unity; refer to Chatterjee's thesis

`http://www.maths.ox.ac.uk/~hitchin/hitchinstudents/chatterjee.pdf`

Recall that the choice of 0-connection can be used to define an exact courant algebroid

$$0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0 \ .$$

Show that the choice of 1-connection defines a splitting $s : T \longrightarrow E$ of this sequence; show that

$$s^*[s(X), s(Y)] = i_X i_Y H,$$

where $s^* : E \longrightarrow T^*$ is the induced projection.

Exercise 3. Let β, β' be gauge equivalent Poisson structures, i.e.

$$\beta' = \beta(1 + B\beta)^{-1}$$

for $B \in \Omega^2_{cl}(M)$ and such that the inverse above exists. Verify that β' is indeed Poisson, and show that there is a canonical isomorphism between the β and β' Poisson cohomology groups.

Exercise 4. Let $\rho \in \Omega^\bullet(G)$ be a left-invariant form on a Lie group G. Show that $d\rho = 0$ if ρ is also right-invariant.

Exercise 5. Let $\theta^L \in \Omega^1(G, \mathfrak{g})$ be the left Maurer-Cartan form. Show that

$$d\theta^L + \tfrac{1}{2}[\theta^L, \theta^L] = 0,$$

where $[\theta, \theta] \in \Omega^2(M, \mathfrak{g})$ is defined by $[\theta, \theta](v, w) = [\theta(v), \theta(w)]$ for any $\theta \in \Omega^1(M, \mathfrak{g})$ and v, w vector fields. Hint: See Helgason "Differential geometry, Lie groups, and symmetric spaces", p. 138.

Exercise 6. Show that $j^*\theta^R = -\theta^L$. Conclude from the above that $d\theta^R - \frac{1}{2}[\theta^R, \theta^R] = 0$. Verify this explicitly for the case $G = GL_n$.

Exercise 7. Show that

$$\theta^R(a^L)|_g = \mathrm{Ad}_g(a)$$

for $a \in \mathfrak{g}$.

Exercise 8. Let B be a bi-invariant metric on the Lie group G. Show that

$$B(\theta^L, [\theta^L, \theta^L])(a^L, b^L, c^L) = 6B(a^L, [b^L, c^L]) = -6B(a^R, [b^R, c^R]).$$

Exercise 9. Let $L_C = \mathrm{Span}\{a^L - a^R + B(a^L + a^R) : \ a \in \mathfrak{g}\}$ be the canonical Dirac structure on (G, B).

- We saw that it is a twisted Poisson structure Γ_β in the open set $U \subset G$ where $\mathrm{Ad}_g + 1 \in \mathrm{End}(\mathfrak{g})$ is invertible. Determine explicitly the bivector β in U.
- For $G = SU(2) = S^3$, describe the conjugacy classes and the loci where $\mathrm{Ad}_g + 1$ is invertible, has rank 2, 1, and 0.

** Determine the Lie algebroid cohomology $H^\bullet(L_C)$. (Hint: the map $\mathfrak{g} \longrightarrow L_C$ given by $a \mapsto a^L - a^R + B(a^L + a^R)$ is bracket-preserving).

Exercise 10. Let $L \subset V \oplus V^*$ be a linear Dirac structure, so that it is determined by $C = L \cap V$ and $\gamma \in \wedge^2(V/C)$. If $f : V \longrightarrow W$ is a linear map, show that the induced forward Dirac map can be expressed as follows:

$$f_* L(C, \gamma) = L(f_* C, f_* \gamma).$$

In particular, conclude that the forward Dirac map induced by a composition of linear maps $f : V \longrightarrow W$, $g : W \longrightarrow X$ satisfies $(g \circ f)_* = g_* \circ f_*$.

State and prove the analogous results for backward Dirac maps.

For an alternative proof of this fact using Weinstein's symplectic category, see "Gauge equivalence of Dirac structures and symplectic groupoids," Bursztyn and Radko, math.SG/0202099.

Exercise 11. Let $L \subset T \oplus T^*$ be an H-twisted Dirac structure on N and let $M \subset N$ be a leaf of the generalized distribution $\Delta = \pi_T(L)$ on N. Then $L = L(\Delta, \epsilon)$ where $\epsilon \in \Omega^2(M)$. Show using the integrability of L that $d\epsilon = f^*H$, where $f : M \longrightarrow N$ is the inclusion map. Hence conclude that (M, ϵ) is a Dirac brane for (N, H, L).