18.969 Topics in Geometry, MIT Fall term, 2006

Problem sheet 5

Exercise 1. Let $\mathcal{J}$ be an even generalized complex structure on a real 4-dimensional manifold. Then the canonical pure spinor line is a sub-bundle

$$K \subset \wedge^\bullet T^* \otimes \mathbb{C}.$$

the projection $s : K \to \wedge^0 T^* \otimes \mathbb{C}$ defines a section $s \in C^\infty(K^*)$. Show that $\overline{\partial} s = 0$, where $\overline{\partial}$ is the generalized Dolbeault operator induced by the generalized complex structure.

Exercise 2. Construct examples of generic even and odd generalized complex structures in dimension 6. That is, they must be of type 0 and 1 almost everywhere, respectively. Construct such examples which fail to be of type 0 or 1 at some point.

Exercise 3. Let (g, I, J) determine a hyperKähler structure, with $K = IJ$. Show that $\omega_J + i\omega_K$ is a holomorphic, nondegenerate $(2,0)$ form. Conclude that $\beta = (\omega_J + i\omega_K)^{-1}$ is a holomorphic Poisson structure. Compute the β-transform of $\mathcal{J}_I$. What is its type?

Exercise 4. Let (J, ω) be a Kähler structure, and let β be a holomorphic Poisson structure, so that $\mathcal{J}_B = e^{tQ}\mathcal{J}_J e^{-tQ}$ is integrable for all t. What is the condition on β which guarantees that $\mathcal{J}_A = e^{tQ}\mathcal{J}_\omega e^{-tQ}$ is integrable for small t? What are the resulting types of $(\mathcal{J}_A, \mathcal{J}_B)$?

Exercise 5. Let $* = a_1 \cdots a_n \in CL(T \oplus T^*)$ be the Hodge star associated to the generalized metric G on the oriented manifold M. Recall that the adjoint of d_H in the Born-Infeld inner product is $d_H^* = *d_H*^{-1}$. Determine the symbol sequence associated to d_H^*, $D_\pm = d_H \pm d_H^*$, and $\Delta_H = D_\pm^2$.

Exercise 6. Describe the symplectic leaves of each of the generalized complex structures in the natural generalized Kähler structure $(\mathcal{J}_A, \mathcal{J}_B)$ on $SU(3)$.

Exercise 7. Are there 2-branes in the β-deformed $\mathbb{C}P^2$ which are not complex curves in $\mathbb{C}P^2$?