18.969 Topics in Geometry, MIT Fall term, 2006

Problem sheet 6

Exercise 1. Recall that a morphism $L \in \mathrm{Hom}(E, F)$ is an *epimorphism* when $M \circ L = N \circ L$ implies $M = N$ for any morphisms M, N from F. Show that, in the split orthogonal category, L is an epimorphism if and only if $\pi_F(L) = F$, where $\pi_F : E \times F \longrightarrow F$ is the projection.

Exercise 2. Prove that the double $\mathcal{D} : \mathbf{Vect} \longrightarrow \Theta$ defined by

$$\mathcal{D}(V) = V \oplus V^*$$
$$\mathcal{D}(f) = \{(v + f^*\eta, f_*v + \eta) \in \overline{\mathcal{D}V} \times \mathcal{D}W \ : \ v \in V, \ \eta \in W^*\},$$

for V, W vector spaces and $f : V \longrightarrow W$ a linear map, is a functor. Also show that $\mathcal{D}(f^*) = \mathcal{D}(f)^*$, where dualisation in Θ simply means that $L^* \in \mathrm{Hom}(F, E)$ is defined by $L^* = \{(f, e) \ : \ (e, f) \in L\}$ for $L \in \mathrm{Hom}(E, F)$.

Exercise 3. Prove that $\mathcal{D}$ preserves epi and mono morphisms.

Exercise 4. For any morphism $L \in \mathrm{Hom}(\mathcal{D}V, \mathcal{D}W)$, let $M = \pi(L) \subset V \oplus W$ and $F \in \wedge^2 M^*$ so that $L = j_* e^F M$, where $j : M \hookrightarrow V \oplus W$. Prove that $L = \mathcal{D}\psi_* \circ e^F \circ \mathcal{D}\varphi^*$, where $\psi : M \longrightarrow W$ and $\varphi : M \longrightarrow V$ are the projections.

Prove furthermore that L is an isomorphism if and only if M projects surjectively onto V and W, and F determines a nondegenerate pairing between $\ker \varphi \subset M$ and $\ker \psi \subset M$.

Exercise 5. What is the T-dual of the trivial S^1 bundle over S^2 with $H = k\nu$ where $k \in \mathbb{Z}$ and ν is the generator of $H^3(S^1 \times S^2, \mathbb{Z})$?

Exercise 6. Verify the Buscher rules $g + b \mapsto \tilde{g} + \tilde{b}$ under a single S^1 T-duality.