1 Lecture 1 (Notes: K. Venkatram)

1.1 Smooth Manifolds

Let M be a f.d. C^∞ manifold, and $C^\infty(M)$ the algebra of smooth $\mathbb{R}$-valued functions. Let $T = TM$ be the tangent bundle of M: then $C^\infty(T)$ is the set of derivations $\mathrm{Der}(C^\infty(M))$, i.e. the set of morphisms $X \in \mathrm{End}(C^\infty(M))$ s.t. $X(fg) = (Xf)g + f(Xg)$. Then $C^\infty(T)$ is equilled with a Lie bracket $[,]$ via the commutator $[X,Y]f = XYf - YXf$.

Note. Explicitly, $[X,Y]$ can be obtained as $\lim_{t\to 0} \frac{Y - \mathrm{Fl}^t_X Y}{t}$, where $\mathrm{Fl}^t_X \in \mathrm{Diff}(M)$ is the *flow* of the vector field on M.

Definition 1. *The* exterior derivative *is the mapping*

$$
\begin{aligned}
d : C^\infty\left(\bigwedge^k T^*\right) &\to C^\infty\left(\bigwedge^{k+1} T^*\right) \\
p \mapsto \Bigg[(X_0,\ldots,X_k) &\mapsto \sum_i (-1)^i X_i p(X_0,\ldots,\hat{X}_i,\ldots,X_k) \\
&+ \sum_{i<j} (-1)^{i+j} p([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k)\Bigg]
\end{aligned} \tag{1}
$$

Since $[,]$ satisfies the Jacobi identity, $d^2 = 0$, i.e.

$$
\cdots \to C^\infty\left(\bigwedge^{k-1} T^*\right) \xrightarrow{d} C^\infty\left(\bigwedge^{k} T^*\right) \xrightarrow{d} C^\infty\left(\bigwedge^{k+1} T^*\right) \to \cdots \tag{2}
$$

is a *differential complex* of first-order differential operators. Set $\Omega^k(M) = C^\infty(\bigwedge^k T^*)$. Letting $m_f = \{g \mapsto fg\}$ denote multiplication by f, one finds that $[d, m_f]\rho = df \wedge \rho$, thus obtaining a sequence of symbols

$$
\bigwedge^{k-1} T^* \xrightarrow{\eta\wedge\cdot} \bigwedge^{k} T^* \xrightarrow{\eta\wedge\cdot} \bigwedge^{k+1} T^* \tag{3}
$$

which is exact for any nonzero 1-form $\eta \in C^\infty(T^*)$. Thus, Ω^* is an *elliptic complex.* In particular, if M is compact, $H^*(M) = \frac{\mathrm{Ker}\ d|_{\Omega^*}}{\mathrm{Im}\ d|_{\Omega^*-1}}$ is finite dimensional.

Remark. d has the property $d(\alpha\wedge\beta) = d\alpha \wedge d\beta + (-1)^{\deg\ \alpha}\alpha\wedge d\beta$. Thus, $(\Omega^\bullet(M), d, \wedge)$ is a differential graded algebra, and $H^\bullet(M) = \bigoplus H^k(M)$ has a ring structure (called the *de Rham cohomology ring*).

We would like to express $[X,Y]$ in terms of d. Now, a vector field $X \in C^\infty(T)$ determines a derivation

$$
i_X : \Omega^k(M) \to \Omega^{k-1}(M), \rho \mapsto [(Y_1,\ldots,Y_k) \mapsto \rho(X,Y_1,\ldots,Y_k)] \tag{4}
$$

of $\Omega^*(M)$. i_X has degree -1 and order 0.

Definition 2. *The* Lie derivative *of a vector field X is $L_X = [i_X, d]$.*

Note that this map has order 1 and degree 0.

Theorem 1 (Cartan's formula)**.** $i_{[X,Y]} = [[i_X, d], i_Y]$

One thus obtains $[,]$ as the *derived bracket of d*. See Kosmann-Schwarzbach's "Derived Brackets" for more information.

Problem. Classifly all derivations of $\Omega^\bullet(M)$, and show that the set of such derivations has the structure of a $\mathbb{Z}$-graded Lie algebra.

One can extend the Lie bracket $[,]$ on vector fields to an operator on all $C^\infty(\bigwedge^k T)$.

Definition 3. *The* Shouten *bracket is the mapping*

$$[,] : C^\infty\left(\bigwedge^p T\right) \times C^\infty\left(\bigwedge^q T\right) \to C^\infty\left(\bigwedge^{p+q-1} T\right)$$
$$(X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q) \mapsto \sum (-1)^{i+j}[X_i, Y_i] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_q \tag{5}$$

with the additional properties $[X, f] = -[f, X] = X(f)$ and $[f, g] = 0 \forall f, g \in C^\infty(M)$.

Note the following properties:

- $[P, Q] = -(-1)^{(\deg P-1)(\deg Q-1)}[Q, P]$
- $[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(\deg P-1)\deg Q} Q \wedge [P, R]$
- $[P, [Q, R]] = [[P, Q], R] + (-1)^{(\deg P-1)(\deg Q-1)}[Q, [P, R]]$

Thus, we find that $C^\infty(\bigwedge T)$ has two operations: a wedge product $\wedge$, giving it the structure of a graded commutative algebra, and a bracket $[,]$, giving it the structure of the Lie algebra. The above properties imply that it is a *Gerstenhaber algebra*.

Finally, for $P = X_1 \wedge \cdots \wedge X_p$, define $i_p = i_{X_1} \circ \cdots \circ i_{X_p}$. Note that it is a map of degree $-p$

Problem. Show that $[[i_P, d]i_Q] = (-1)^{(\deg P-1)(\deg Q-1)} i_{[P,Q]}$.

1.2 Geometry of Foliations

Let $\Delta \subset T$ be subbundle of the tangent bundle (*distribution*) with constant rank k.

Definition 4. *An* integrating foliation *is a decomposition $M = \bigsqcup S$ of M into "leaves" which are locally embedded submanifolds with $TS = \Delta$.*

Note that such leaves all have dimension k.

Theorem 2 (Frobenius)**.** *An integrating foliation exists $\Leftrightarrow$ Δ is* involutive*, i.e.* $[\Delta, \Delta] \subset [\Delta]$.

A distribution is equivallently determined by Ann $\Delta \subset T^*$ or the line det Ann $\Delta \subset \Omega^{n-k}(M)$. That is, for locally-defined 1-forms $(\theta_1, \ldots, \theta_{n-k})$ s.t. $\Delta = \bigcap_i \text{Ker } \theta_i$, $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ generates a line bundle. If Δ is involutive, $i_X i_Y d\Omega = [[i_X, d], i_Y]\Omega = i_{[X,Y]}\Omega = 0$ for all X, Y s.t. $i_X\Omega = i_Y\Omega = 0$. That is, $d\Omega = \eta \wedge \Omega$ for some 1-form $\eta \in \Omega$.

Remark. More generally, let $\Delta \subset T$ be a distribution on non-constant rank spanned by an nvolutive $C^\infty(M)$ module $\mathcal{D} \subset C^\infty(T)$ at each point. Sussmann showed that such a $\mathcal{D}$ gives M as a disjoint union of locally embedding leaves S with $TS = \Delta$ everywhere.

1.3 Symplectic Structure

Definition 5. *An* symplectic structure *on M is a closed, non-degenerate two-form $\omega : T \to T^*$.*

Let (M, ω) be a symplectic manifold: note that $\det \omega \in \det T^* \otimes \det T^*$.

Problem. Show that $\det \omega = \text{Pf } \omega \otimes \text{Pf } \omega$, where Pf is the *Pfaffian.*

Theorem 3 (Darboux)**.** *Locally, $\exists C^\infty$ functions $p_1, \ldots, p_n, q_1, \ldots, q_n$ s.t. $\{dp_i, dq_i\}$ span T^* and $\omega = \sum dp_i \wedge dq_i$. That is, (M, ω) is locally diffeomorphic to $(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$.*

Moreover, by Stokes' theorem, one finds that $\int_M \omega \wedge \cdots \wedge \omega \neq 0 \implies [\omega]^i \neq 0$ for all i.

Corollary 1. *Neither S^4 nor $S^1 \times S^3$ have a symplectic structure.*