11 Lecture 11(Notes: K. Venkatram)

11.1 Integrability and spinors

Given $L \subset T \oplus T^*$ maximal isotropic, we get a filtration $0 \subset K_L = F^0 \subset F^1 \subset \cdots \subset F^n = \Omega^*(M)$ via $F^k = \{\psi : \bigwedge^{k+1} L \cdot \psi = 0\}$. Furthermore, for $\phi \in K_L$, we have

$$X_1 X_2 d\phi = [[d, X_1], X_2]\phi = [X_1, X_2]\phi \tag{21}$$

for all $X_1, X_2 \in L$ (where $d = d_H$). Thus, in general, $d\phi \in F^3$, and L is *involutive* $\Leftrightarrow d\phi \in F^1$.
Now, assume $d(F^i) \subset F^{i+3}$ (and in F^{i+1} if L is integrable) $\forall i < k$ and $\psi \in F^k$. Then

$$\begin{aligned} [X_1, X_2]\psi = [[d, X_1], X_2]\psi = dX_1X_2\psi + X_1dX_2\psi - X_2dX_1\psi - X_2X_1d\psi \\ X_1X_2d\psi = -dX_1X_2\psi - X_1dX_2\psi + X_2dX_1\psi + [X_1, X_2]\psi \end{aligned} \tag{22}$$

Note that, in the latter expression, each of the parts on the RHS have degree $(k-1)+2 = k+1$, so $d\psi \in F^{k+1}$ if L is integrable and F^{k+3} otherwise.
Next, suppose that the Courant algebroid E has a decomposition $L \oplus L'$ into transverse Dirac structures.

1. Linear algebra:

   - $L' \cong L^*$ via $\langle \cdot, \cdot \rangle$.
   - The filtration $K_L = F^0 \subset F^1 \subset \cdots \subset F^n$ of spinors becomes a $\mathbb{Z}$-grading $K_L \oplus (L' \cdot K_L) \oplus \cdots \oplus (\bigwedge^k L' \cdot K_L) \oplus \cdots \oplus (\det L' \cdot K_L)$, i.e. $\bigoplus (\bigwedge^k L^*) K_L$.

     Remark. Note that $L' \cdot (\det L' \cdot K_L) = 0$, so $\det L' \cdot K_L = \det L^* \otimes K_L = K_{L'}$.

     Thus, we have a $\mathbb{Z}$ grading $S = \bigoplus_{k=0}^n \mathcal{U}_k$.
   - If the Mukai pairing is nondegenerate on pure spinors, then $K_L \otimes K_{L'} = \det T^*$.

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2. Differential structure: via the above grading, we have $F^k(L) = \bigoplus_{i=0}^{k} \mathcal{U}_i, F^k(L') = \bigoplus_{i=0}^{k} \mathcal{U}_{n-i}$, so $d(\mathcal{U}_k) = d(F^k(L) \cap F^{n-k}(L'))$. By parity, $d\mathcal{U}_k \cap \mathcal{U}_k = 0$, so a priori

$$d = (\pi_{k-3} + \pi_{k-1} + \pi_{k+1} + \pi_{k+3}) \circ d = T' + \partial' + \partial + T \tag{23}$$

Problem. Show that $T' : \mathcal{U}_k \to \mathcal{U}_{k-3}, T : \mathcal{U}_k \to \mathcal{U}_{k+3}$ are given by the Clifford action of tensors $T' \in \bigwedge^3 L, T \in \bigwedge^3 L^*$.

Remark. This splitting of $d = d_H$ can be used to understand the splitting of the Courant structure on $L \oplus L^*$. Specifically, $d^2 = 0 \implies$

$$\begin{array}{cc} -4 & T'\partial' + \partial' T' = 0 \\ -2 & (\partial')^2 + T'\partial + \partial T' \\ 0 & \partial\partial' + \partial'\partial + TT' + T'T \\ 2 & \partial^2 + T\partial' + \partial' T \\ 4 & T\partial + \partial T = 0 \end{array} \tag{24}$$

11.2 Lie Bialgebroids and deformations

We can express the whole Courant structure in terms of (L, L^*). Assume for simplicity that L, L^* are both integrable, so $T = T' = 0$. Then

1. Anchor $\pi \to$ a pair of anchors $\pi : L \to T, \pi' : L' \to T$.
2. An inner product $\to$ a pairing $L' = L^*, \langle X + \xi, X + \xi \rangle = \xi(X)$.
3. A bracket $\to$ a bracket $[,]$ on L, $[,]_*$ on L^*. Specifically, for $x, y \in L, \phi \in \mathcal{U}_0$,

$$[x, y]\phi = [[d, x], y]\phi = xyd\phi = xy(\partial + T)\phi = xyT\phi = (i_x i_y T)\phi \tag{25}$$

The induced action on S is $d_L\alpha = [\partial, \alpha]$, giving us an action of L on L^* as $\pi_{L^*}[x, \xi]$ for $x \in L, \xi \in L^*$. Expanding, we have

$$\begin{aligned} [x, \xi]\phi &= [[\partial, x], \xi]\phi = \partial x\xi\phi + x\partial\xi\phi - \xi x\partial\phi - (i_x\xi)\partial\phi \\ &= \partial(i_x\xi)\phi + x(d_L\xi)\phi - (i_x\xi)\partial\phi = (d_L i_x\xi + i_x d_L\xi)\phi = (L_x\xi)\phi \end{aligned} \tag{26}$$

If $T = 0$, then $x \to L_x$ is an action (guaranteed by the Jacobi identity of the Courant algebroid). If L, L' are integrable,

$$L_x[\xi, \eta]_* = \pi_{L^*}[x, [\xi, \eta]] = \pi_{L^*}([[x, \xi], \eta] + [\xi, [x, \eta]]) \tag{27}$$

Problem. This implies that $d[\cdot, \cdot]_* = [d\cdot, \cdot]_* + [\cdot, d\cdot]_*$.

As a result of these computations, we find that, for $X, Y \in L, \xi, \eta \in L^*$,

$$\begin{aligned} [X + \xi, Y + \eta] &= [X, Y] + [X, \eta]_L + [\xi, Y]_L + [\xi, \eta] + [\xi, Y]_{L^*} + [X, \eta]_{L^*} \\ &= [X, Y] + L_\xi Y - i_\eta d_* X + [\xi, \eta] + L_X\eta - i_Y d\xi \end{aligned} \tag{28}$$

There are no H terms since we assumed $T = T' = 0$. Overall, we have obtained a correspondence between transverse Dirac structures (L, L') and Lie bialgebroids (L, L^*) with actions and brackets $L \to T, L^* \to T$ s.t. d is a derivation of $[,]_*$.

Finally, we can deform the Dirac structure in pairs. Specifically, for $\epsilon \in C^\infty(\bigwedge^2 L^*)$ a small B-transform, $e^\epsilon(L) = L_\epsilon$, one can ask when L_ϵ is integrable. We claim that this happens $\Leftrightarrow d_L\epsilon + \frac{1}{2}[\epsilon, \epsilon]_* = 0$. To see this, note that

$$\begin{aligned} \langle [e^\epsilon x, e^\epsilon y], e^\epsilon z\rangle &= \langle [e^\epsilon x, e^\epsilon y]_L, e^\epsilon z\rangle + \langle [e^\epsilon x, e^\epsilon y]_{L^*}, e^\epsilon z\rangle \\ &= (d_L\epsilon)(x, y, z) + \frac{1}{2}[\epsilon, \epsilon]_*(x, y, z) \end{aligned} \tag{29}$$

via an analogous computation to that of $e^B T$ and $e^\pi T^*$ from before.