12 Lecture 12-17(Notes: K. Venkatram)

12.1 Generalized Complex Structures and Topological Obstructions

Let $E \cong (T \oplus T^*, H)$ be an exact Courant algebroid.

Definition 19. *A* generalized complex structure *(GCS) on E is an integrable orthogonal complex structure $\mathbb{J} : E \to E$, i.e. a map s.t.*

- $\langle \mathbb{J}A, \mathbb{J}B \rangle = \langle A, B \rangle$
- $L = \text{Ker } (\mathbb{J} - i1)$

Note. 1. $\langle \mathbb{J}A, B \rangle = \langle \mathbb{J}^2 A, \mathbb{J}B \rangle = -\langle A, \mathbb{J}B \rangle$, and thus $\langle \mathbb{J}\cdot, \cdot \rangle$ is a symplectic struction on E compatible with $\langle , \rangle$.

2. L is maximal isotropic and so is $\overline{L}$, and thus $E = L \oplus \overline{L} = L \oplus L^*$ and we get a Lie bialgebroid.

3. V must be even dimensional: letting $x \in V \oplus V^*$ be a null vector then $\langle \mathbb{J}x, x \rangle = 0$ and $\langle \mathbb{J}x, \mathbb{J}x \rangle = 0$, so we can always enlarge a null set by 2 vectors; thus the maximal null set is even.

At the level of structure groups, $(T \oplus T^*, \langle , \rangle), \mathbb{J}$ corresponds to
$O(2n, 2n) \to U(n, n) = O(2n, 2n) \cap GL(2n, \mathbb{C})$.

Problem. Show that $O(V \oplus V^*)$ acts transitively by conjugation on a set of GCS

$$S_{\mathbb{J}} \cong \frac{O(2n, 2n)}{U(n, n)} \tag{30}$$

Example. 1. $\mathbb{J} = \begin{pmatrix} J & \\ & -J^* \end{pmatrix}$ acting on $V \oplus V^*$.

2. $\mathbb{J} = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$ acting on $V \oplus V^*$.

3. Any conjugation $A\mathbb{J}A^{-1}, A \in O(2n, 2n)$, e.g. $e\mathbb{J}e^{-1}$,

$$\begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \begin{pmatrix} J & \\ & -J^* \end{pmatrix} \begin{pmatrix} 1 & \\ -B & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \begin{pmatrix} J & 0 \\ J^*B & -J^* \end{pmatrix} = \begin{pmatrix} J & 0 \\ J^*B + BJ & -J^* \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} \begin{pmatrix} 1 & \\ -B & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} \multicolumn{2}{c}{\omega^{-1}B} \\ \multicolumn{2}{c}{-\omega^{-1}} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix} \tag{31}$$

Lemma 3. $O(n.n) \simeq O(n) \times O(n)$.

Proof. Let $C_+ \subset V \oplus V^*$ be positive definite and $C_- = C_+^\perp$. THen $O(n,n)$ acts transitively on the space of all C_+, with stabilizer $\mathrm{Stab}(C_+) = O(n) \times O(n)$. Question: what is $\frac{O(n,n)}{O(n)\times O(n)}$? C_+' (see diagram below) is given by $A : \mathbb{R}^n \to \mathbb{R}^n, ||Ax|| < ||x|| \forall x$, i.e. $||A||_{op} < 1$. Thus, it is the unit ball under the operator norm. □

Lemma 4. $U(n,n) \simeq U(n) \times U(n)$

Proof. We can enlarge $\tilde{C}_+$ to C_+ by adding $V \perp \tilde{C}_+$ and $\mathbb{J}V$, and get complex decomposition $E = C_+ \oplus C_+^\perp = C_+ + C_-$. $U(n,n)$ acts transitively on these spaces with stabilizer $\mathrm{Stab}(C_+) = U(n) \times U(n)$. As above, we obtain the unit ball in $\mathbb{C}^n$. □

Thus, the existence of $\mathbb{J}$ is topologically equivalent to the reduction to $U(n) \times U(n)$, i.e. complex structures $\mathbb{J}_\pm := \mathbb{J}|_{C_\pm}$ on C_+ and $C_- = C_+^\perp$ (since the bundle of positive-definite subspaces is contractible).

Note. The projection $\pi : C_\pm \to T$ is an isomorphism, so we obtain almost complex structure $J_\pm : T \to T$.

Thus M must be almost complex, and $\mathbb{J}$ has two sets of Chern classes $c_i^\pm \in H^{2i}(M, \mathbb{Z})$ associated to $J_\pm$ (i.e. $c_i^\pm = c_i(c_\pm)$) and $c(T \oplus T^*, \mathbb{J}) = c(C_+) \cup c(C_-)$.

Remark. Topologically, E has structure group $U(n,n) \simeq U(n) \times U(n)$, so the bundle is classified by $\psi : X \to B(U(n) \times U(n)) = BU(n) \times BU(n) = C^+ \times C^-$ with Chern classes $\psi^* C^+, \psi^* C^-$.

Now, spaces $L \subset T \oplus T^*$ correspond to canonical bundes $K_L \subset \Omega^*(M)$.

Proposition 5. *A generalized complex structure is equivalent to a complex Dirac structure of real index 0, i.e. to a Dirac structure* $L \subset (T \oplus T^*) \otimes \mathbb{C}$ *s.t.* $\overline{L} \cap L = \{0\}$.

Proof. $\Leftarrow$: given L, set $\mathbb{J} = i|_L + (-i)|_{\overline{L}}$, and obtain

$$\langle \mathbb{J}(\alpha + \overline{\beta}), \mathbb{J}(\alpha + \overline{\beta}) \rangle = \langle i\alpha - i\overline{\beta}, i\alpha - i\overline{\beta} \rangle = \langle \alpha, \overline{\beta} \rangle + \langle \overline{\beta}, \alpha \rangle = \langle \alpha + \overline{\beta}, \alpha + \overline{\beta} \rangle \tag{32}$$

$\to$: given $\mathbb{J}$, set $L = \mathrm{Ker}\ (\mathbb{J} - i1)$, so

$$\langle \alpha, \beta \rangle = \langle \mathbb{J}\alpha, \mathbb{J}\beta \rangle = -\langle \alpha, \beta \rangle = 0 \tag{33}$$

□

Therefore, $(T \oplus T^*) \otimes \mathbb{C} = L \oplus \overline{L}$, and we obtain a transverse complex Dirac structure. This gives us a $\mathbb{Z}$-grading on $S \otimes \mathbb{C} = \Omega^*(M, \mathbb{C})$ as

$$(K_L = \mathcal{U}_n) \oplus \mathcal{U}_{n-1} \oplus \cdots \oplus \mathcal{U}_{-n+1} \oplus (\mathcal{U}_{-n} = K_{\overline{L}}) \tag{34}$$

with conjugation exchanging $\mathcal{U}_k$ and $\mathcal{U}_{-k}$.

Definition 20. $K_L = \mathcal{U}_{-n}$ *is the canonical line bundle of the generalized complex structure.*

Furthermore, the decomposition $d_H = \partial + \overline{\partial}$ gives the general Dolbeault complex via $\partial : \mathcal{U}_k \leftrightarrow \mathcal{U}_{k-1} : \overline{\partial}$.

Problem. Use the Mukai pairing between K_L and $\overline{K}_L$ to show that $2c_1(K_L) = c_1^* + c_1^-$.

12.1.1 $\mathbb{Z}$-grading on spinors

Let $\mathbb{J}$ be a generalized complex structure: then $\mathbb{J} \in \mathfrak{so}(T \oplus T^*)$. The transformtation $e^{\theta\mathbb{J}}$ behaves like $e^{i\theta}$ and thus defines an S^1 action on $T \oplus T^*$ and thus, by the spin representation, on on $\Omega^*(M)$ (in fact, we can imagine this as $\cos\theta \cdot 1 + \mathbb{J} \cdot \sin\theta$). Just as $(T \oplus T^*) \otimes \mathbb{C}$ decomposes as $L \oplus \overline{L}$, we have $\mathbb{J}(x, \phi) = [\mathbb{J}, x] \cdot \phi + x \cdot \mathbb{J}\phi$, where $[\mathbb{J}, x]$ is the $\mathfrak{so}$-action. Thus, for an eigenvector $x \in L, \mathbb{J}x = ix$, then $\mathbb{J}x\phi = x\mathbb{J}\phi + i\phi$. That is, the action of L increases by i, while $\overline{L}$ decreases by i, givng us a diagram

$$K_{\overline{L}} = \mathcal{U}_{-n} \underset{\overline{L}}{\overset{L}{\rightleftarrows}} \mathcal{U}_{-n+1} \qquad \cdots \qquad \mathcal{U}_{n-1} \underset{\overline{L}}{\overset{L}{\rightleftarrows}} \mathcal{U}_n = K_L \tag{35}$$

Since the eigenvalues are symmetric, they must be $\{-ni, (-n+1)i, \dots, ni\}$, with $\mathcal{U}_k$ the ik-eigenspace of $\mathbb{J}$. Now, via the decomposition $d_H = \partial + \overline{\partial}$, we can form another real differential operator $d^{\mathbb{J}} = [d, \mathbb{J}] = [\partial + \overline{\partial}, \mathbb{J}]$. Applying this to ϕ^k gives

$$[d, \mathbb{J}]\phi^k = ik(\partial + \overline{\partial})\phi - i(k+1)\partial\phi - i(k-1)\overline{\partial}\phi = i(\overline{\partial} - \partial)\phi \tag{36}$$

Thus, $d^{\mathbb{J}} = i(\overline{\partial} - \partial)$, and $(d^{\mathbb{J}})^2 = 0$ as desired.
For each GCS, we obtain three complexes: $(C^\infty(\bigwedge^* L^*), d_L)$ and the pair $(\mathcal{U}^*, \overline{\partial}), (\mathcal{U}^*, \partial)$.

Proposition 6. *$(C^\infty(\bigwedge^* L^*), d_L)$ is elliptic.*

Recall that in general, this is not true. In particular, in the case of Poisson structures, the complex is infinite dimensional.

Proof. Since L is a Lie algebra, we obtain a symbol sequence

$$\bigwedge^{k-1} L^* \to^{S_\xi} \bigwedge^{k} L^* \to^{S_\xi} \bigwedge^{k+1} L^* \tag{37}$$

where $S_\xi(\phi) = \pi^*\xi \wedge \phi$ for a given $\xi \in T^*$ real. If $\xi \neq 0$, it can be decomposed as $\alpha + \overline{\alpha} \in L \oplus \overline{L}$ with $\alpha \neq 0$. Moreover, for $x \in L$, we have

$$(\pi^*\xi)(x) = \xi(\pi x) = \langle \xi, x \rangle = \langle \alpha + \overline{\alpha}, x \rangle = \langle \overline{\alpha}, x \rangle \tag{38}$$

so $\pi^*\xi = \overline{\alpha}$ is nonzero. $\square$

Corollary 4. *$H^*(L), H^*(\overline{L})$ are finite dimensional on compact generalized complex manifolds.*

For the other complex, we have that $d_H(f\phi) = df \wedge \phi + f d_H\phi = (d_L f + d_{\overline{L}} f)\phi + f d_H \phi$, so that $\overline{\partial}(f\phi) = (d_L f)\phi + f\overline{\partial}\phi$.

Problem. Using the right derived bracket, show that $(d_L x)\cdot = [\overline{\partial}, x\cdot]$ for $x \in C^\infty(\bigwedge^k L^*)$.

By the above, we have a symbol sequence $\mathcal{U}^{k-1} \leftarrow^{S_\xi} \mathcal{U}^k \leftarrow^{S_\xi} \mathcal{U}^{k+1}$ given by the anihilation operator $S_\xi(\phi) = \overline{\alpha}\phi$ which is also an exact sequence. Doing a similar procedure for ∂, and following the above logic (replacing the Clifford action with the wedge product), we obtain:

Corollary 5. *$H^*_{\overline{\partial}}(M), H^*_{\partial}(M)$ are finite dimensional for compact generalized complex manifolds.*

Remark. One has a spectral sequence $H^*_{\partial,\overline{\partial}}(M) \implies H^*_{d_H}(M)$. Moreover, this spectral sequence is trivial (i.e. $H^*_{d_H} = \bigoplus H^*_{\overline{\partial}}(M)$ if the $\partial\overline{\partial}$-lemma holds for M: if $\overline{\partial}\alpha = 0$ and $\alpha = \partial\beta$, then $\alpha = \overline{\partial}\partial\gamma$ for some γ. In other words,

$$\text{Im } \partial \cap \text{Ker } \overline{\partial} = \text{Ker } \partial \cap \text{Im } \overline{\partial} = \text{Im } \partial\overline{\partial} \tag{39}$$

Finally, we obtain actions of $H^*(L), H^*(\overline{L})$ on $H^*_{\overline{\partial}}(M), H^*_{\partial}(M)$ respectively via

$$\overline{\partial}(x \cdot \phi) = (d_L x) \cdot \phi + (-1)^x x \cdot \overline{\partial}\phi, x \in \bigwedge^k L \tag{40}$$

Problem. Show the above statement.

This statement implies $d_L x = [\overline{\partial}, x]$, so $d_L x = 0, \overline{\partial}\phi = 0 \implies \overline{\partial}(x \cdot \phi) = 0$, making the action well-defined.

12.1.2 Complex Case

Given an almost-complex structure J, we obtain a generalized complex structure $\mathbb{J}_J = \begin{pmatrix} -J & \\ & J^* \end{pmatrix}$. We claim that $\mathbb{J}_J$ is integrable w.r.t. $[,] \Leftrightarrow J$ is integrable. To see this, decompose $L = T_{0,1} \oplus T^*_{1,0}$, and choose elements $x, y \in T_{0,1}, \xi, \eta \in T^*_{1,0}$. One obtains

$$[x, y] + L_x\eta - i_y d\xi = [x, y] + i_x\overline{\partial}\eta - i_y\overline{\partial}\xi \tag{41}$$

where $[x, y] \in T_{0,1} \Leftrightarrow J$ is integrable, and $L_x\eta = i_x d\eta = i_x(\partial\eta + \overline{\partial}\eta) = i_x\overline{\partial}\eta$ because $\partial\eta \in \bigwedge^2 T^*_{0,1}$ and thus does not survive i_x.

Remark. Adding a term $i_x i_y H$ to the above expression, where $H \neq 0$, we find that $i_x i_y H \in T_{1,0} \forall x, y \in T_{0,1} \Leftrightarrow H^{(0,3)} = 0$, i.e. the gerbe is homogeneous. This is similar to the fact that $F^{(2,0)} = 0$ for (L, ∇) holomorphic.

We have two different complexes:

1. First, the complex $(C^\infty(\bigwedge^* L^*), d_L)$, where

$$\bigwedge^k L^* = \bigoplus_{p+q=k} (\bigwedge^p T_{1,0}) \otimes (\bigwedge^p T^*_{0,1}) \tag{42}$$

   and the differential map is given by the individual partials

$$\overline{\partial} : C^\infty(\bigwedge^p T_{1,0} \otimes \bigwedge^p T^*_{0,1}) \to C^\infty(\bigwedge^p T_{1,0} \otimes \bigwedge^{p+1} T^*_{0,1}) \tag{43}$$

   That is, each of the bundles $\bigwedge^p T_{1,0}$ has a $\overline{\partial}$ operator and d_L is their sum. This implies that

$$H^k(L) = \bigoplus_{p+q=k} H^q(\bigwedge^p T_{1,0}) = H^0(\wedge^k T_{1,0}) \oplus H^1(\bigwedge^{k-1} T_{1,0}) \oplus \cdots \oplus H^k(\varnothing) \tag{44}$$

2. Second, we have the complex $(\mathcal{U}^k, \overline{\partial})$ as defined above. Note first that, being the canonical bundles, we have that $K_L = \mathcal{U}^n = \bigwedge^n T^*_{1,0} = \Omega^{n,0}$ (similarly, $K_{\overline{L}} = \mathcal{U}^{-n} = \Omega^{n,0}$. By the decomposition $L = T_{0,1} + T^*_{1,0}$, we find that L acts on each $\Omega^{k,l}$ by either increasing k or decreasing l, giving us our sequence as the decomposed Hodge diamond

$$K_{\overline{L}} = \Omega^{0,n} \left| \begin{matrix} \Omega^{0,n-1} \\ \Omega^{1,n} \end{matrix} \right| \cdots \left| \begin{matrix} \Omega^{0,0} \\ \vdots \\ \Omega^{n,n} \end{matrix} \right| \cdots \left| \begin{matrix} \Omega^{n-1,0} \\ \Omega^{n,1} \end{matrix} \right| \Omega^{n,0} = K_L \tag{45}$$

   That is, $\mathcal{U}^k = \bigoplus_{p-q=k} \Omega^{p,q}$, with the boundary maps given by the usual ones on Ω and $H^k_{\overline{\partial}}(M) = \bigoplus_{p-q=k} H^k_{\nabla d_{\overline{\partial}}}(M)$.

12.1.3 Symplectic Case

Given a symplectic form ω, we obtain a generalized complex structure $\mathbb{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$. Given an i-eigenvector $\begin{pmatrix} x \\ \xi \end{pmatrix}$, we have

$$\omega(x) - \omega^{-1}(\xi) = ix + i\xi \implies i\eta = \omega(x) \tag{46}$$

Thus, $L = \{x - i\omega(x) : x \in T \otimes \mathbb{C}\} = \Gamma_{-i\omega}$, where $\Gamma_{-i\omega}$ denotes the graph of $-i\omega : T \otimes \mathbb{C} \to T^* \otimes \mathbb{C}$, is a simple Dirac structure. Moreover, Ω_σ is integrable w.r.t. $[,]_H \Leftrightarrow d_H\sigma = 0$. In our case, we have $d(-\omega) = -H \wedge (-i\omega)$, so $d\omega$ and H must be 0 (i.e. ω is symplectic). We again get two complexes

1. $(C^\infty(\bigwedge^* L^*) d_L) \cong (C^\infty(\bigwedge^* T^* \otimes \mathbb{C}), d)$ is trivial, and $H^k(L) \cong H^k_{dR}(M)$. However, one does have a nontrivial Gerstenhaber structure $(C^\infty(\bigwedge^* L^*), d_L, [,]_*)$, and one has an equivalence between $(L, \overline{L})$ and $(T \otimes \mathbb{C}, \Gamma_{(\partial i\omega)^{-1}})$ (the Lie bialgebroid of a complex Poisson structure).

2. The ends of the complex $(\mathcal{U}^k, \overline{\partial})$ can be simply exhibited as $K_L = \langle e^{i\omega}\rangle, K_{\overline{L}} = \langle e^{-i\omega}\rangle$. The next term can be computed via

$$\mathcal{U}^{-n+1} = (X - i\omega X)e^{-i\omega} = -i\omega(x) \wedge e^{-i\omega} - i\omega(x) \wedge e^{-i\omega} = e^{i\omega} \cdot \Omega^1 \tag{47}$$

The higher terms are more complicated: given general invertible σ, the transformation $e^{-\sigma} e^{\frac{\sigma^{-1}}{2}}$ on $T \oplus T^*$ sends $T^* \to \Gamma_\sigma$ (i.e. $1 \to e^\sigma$) and $T \to \Gamma_{-\sigma}$ (i.e. $\Omega^n \to e^{-\sigma}$). Thus, we find that

$$\mathcal{U}^k = e^{i\omega} e^{\frac{\omega^{-1}}{2i}} \Omega^{n-k} \tag{48}$$

Letting L, Λ denore the maps $\phi \mapsto \omega \wedge \phi, \phi \mapsto= -i_{\omega^{-1}}\phi$, we obtain the expression $\mathcal{U}^k = e^{iL} e^{-\frac{\Lambda}{2i}} \Omega^{n-k}$. These maps arise via the decomposition of $\mathbb{J}$ as $\begin{pmatrix} & \\ -\omega & \end{pmatrix} + \begin{pmatrix} & \omega^{-1} \\ & \end{pmatrix}$. Setting

$$H = [L, \Lambda] = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{49}$$

we find that $[H, L] = -2L$ and $[H, \Lambda] = 2\Lambda$. These are precisely the $\mathfrak{sl}_2\mathbb{R}$ commutator relations, giving us associated actions on the symplectic manifold. In particular, H acts as

$$H\phi = \frac{1}{2}\mathrm{tr}(\mathrm{id}) - (\mathrm{id}^*)\phi = \mathrm{sum}(n-k)\pi_k\phi \tag{50}$$

where $\pi_k : \Omega \to \Omega^k$ is the projection. Via our decomposition of $\mathbb{J}$, we find that $d^{\mathbb{J}} = [d, L + \Lambda] = [d, \Lambda] = \delta$ is a degree -1 operator with $\delta^2 = 0$ (called the *symplectic adjoint of* d) and $\overline{\partial} = d - i\delta : \mathcal{U}^k \to \mathcal{U}^{k-1}$. Using an analogous $d\delta$ (or $\partial\overline{\partial}$) lemma for symplectic manifolds, we find that any cohomology class $\alpha \in H^*_{dR}$ has a δ-closed representation (since $\delta\alpha = \delta d\gamma$ and $d(\alpha - \gamma) = 0$, implying that $\delta(\alpha - d\gamma) = 0$). Thus, setting $\tilde{\alpha} = \alpha - \gamma$, we find that $[d, \mathbb{J}]\widetilde{alpha} = 0 \Leftrightarrow [d, \Lambda]\tilde{\alpha} = 0 \implies d(\bigwedge \tilde{\alpha}) = 0$. These statements combine to give an action of (L, Λ) on cohomology, i.e. an $\mathfrak{sl}_2\mathbb{R}$ action on $H^*(M)$. Furthermore, $L^{n-k} : H^k \to H^{2n-k}$ is an isomorphism, implying an equivalence between the $d\delta$-lemma and the *Lefshetz properlty* (see Cavalcanti thesis for $\Leftarrow$).

12.2 Intermediate Cases

We have studied

$$\mathbb{J}_J = \begin{pmatrix} J & \\ & -J^* \end{pmatrix}, \mathbb{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} \tag{51}$$

What about the intermediate cases?

- intermediate types and spinors
- Poisson structure
- Local form
- Examples of type jumping by deformation
- interpolation

Given a complex bundle $T^* \to E \to^\pi T$, let $\mathbb{J} \circlearrowright E$ with $\mathbb{J}T^* = T^*$. Then $T^* \subset E$ is a complex subspace, and $E/T^* = T$ obtains an almost complex structure J which is integrable. Furthermore,

$$(\mathbb{J}\xi)(X) = \langle \mathbb{J}\xi, \tilde{X} \rangle = -\langle \xi, \mathbb{J}\tilde{x} \rangle = \xi(Jx) = -J^*\xi(X) \tag{52}$$

i.e. $\mathbb{J}|_{T^*} = -J^*$.

12.2.1 Complex and Symplectic Decompositions

Let $S : T \to E$ be any splitting, i.e. $\pi \circ s = id|_T$. Then we can produce a complex splitting by averaging

$$\frac{1}{2}(S - \mathbb{J}sJ) = S' \tag{53}$$

Note. $\pi(-\mathbb{J}sJ)(X) = \pi(-\mathbb{J}(s(JX))) = -J^2X = X$, so $-\mathbb{J}xJ$ is a splitting.

Observe that, in splitting $S' : E \to T \oplus T^*$, we obtain $\mathbb{J} = \begin{pmatrix} J & \\ & -J^* \end{pmatrix}$.

Problem. Write $\mathbb{J}$ is a non-complex splitting using S. Hint: what is the difference between the splittings S and $-\mathbb{J}SJ$?

Finally, assume that $\mathbb{J}T^* \cap T^* = \{0\}$. Then $E = T^* \oplus \mathbb{J}T^*$ and, in this splitting,

$$\mathbb{J} = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} \tag{54}$$

where $\omega(X, Y) = \langle \mathbb{J}xX, xY \rangle$.

12.2.2 General case

In general, $T^* + \mathbb{J}T^*$ is a complex subspace of E, as is $T^* \cap \mathbb{J}^*T^* \subset T^* + \mathbb{J}T^* \subset E$.

Definition 21. $\Delta = \pi(T^* + \mathbb{J}T^*) = \pi\mathbb{J}T^*$.

Note that

$$\text{Ann } \Delta = (T^* + \mathbb{J}T^*)^\perp \cap T^* = T^* \cap \mathbb{J}T^* \cap T^* = T^* \cap \mathbb{J}T^* \tag{55}$$

is complex, and $\frac{T^* + \mathbb{J}T^*}{\text{Ann } \Delta} \cong \Delta^* \oplus_{\circlearrowright \mathbb{J}} \Delta$ has symplectic structure. Also, $E/(T^* + \mathbb{J}T^*) = T/\Delta$ has a complex structure, with complex dimension k (called the *type*).

Theorem 8. *M is generally foliated by symplectic leaves with transverse complex structure.*

Lemma 5. *$\mathbb{J}T^*$ is Dirac.*

Proof. Observe first that the $+i$ eigenspace is closed, i.e.

$$\begin{aligned} z - i\mathbb{J}z &= [x - i\mathbb{J}x, y - i\mathbb{J}y] \\ &= [x,y] - [\mathbb{J}x, \mathbb{J}y] - i([x,\mathbb{J}y] + [\mathbb{J}x, y]) \\ [x,\mathbb{J}y] + [\mathbb{J}x,y] &= \mathbb{J}[x,y] - \mathbb{J}[\mathbb{J}x,\mathbb{J}y] \\ [\mathbb{J}x,\mathbb{J}y] &= [x,y] + \mathbb{J}([x,\mathbb{J}y] + [\mathbb{J}x,y]) \end{aligned} \tag{56}$$

Thus, $[\mathbb{J}\xi,\mathbb{J}\eta] = [\xi,\eta] + \mathbb{J}([\xi,\mathbb{J}\eta] + [\mathbb{J}\xi,\eta]) = \mathbb{J}\alpha$ (note that $\pi\alpha = 0 \implies \alpha \in T^*$). ☐

Problem. Show that $N_{\mathbb{J}}(x,y) = [\mathbb{J}x,\mathbb{J}y] - \mathbb{J}[x,\mathbb{J}y] - \mathbb{J}[\mathbb{J}x,y] - [x,y]$ is tensorial and express it in terms of T, T'.

Problem. $e^{\theta\mathbb{J}}T^*$ is Dirac $\forall\theta$. Hint: $e^{\theta\mathbb{J}}T^* = ((\cos\theta\cdot 1) + (\sin\theta)\mathbb{J})(T^*) = (1+\tan\theta\mathbb{J})T^*$, and

$$[\xi + t\mathbb{J}\xi, \eta + t\mathbb{J}\eta] = t([\xi,\mathbb{J}\eta] + [J\xi,\eta]) + t^2\mathbb{J}([\xi,\mathbb{J}\eta] + [\mathbb{J}\xi,\eta]) = (1+t\mathbb{J})(t([\xi,\mathbb{J}\eta] + [\mathbb{J}\xi,\eta])) \tag{57}$$

Lemma 6. *For small θ, $e^{\theta\mathbb{J}}T^*$ is a twisted Poisson structure in a splitting satisfying $[\pi,\pi] = \bigwedge^3 \pi^* H$.*

Taking the derivative $\frac{d}{d\theta}(e^{\theta\mathbb{J}}T^*)$ at $\theta = 0$, we obtain a tangent vector to $\mathrm{Dir}(T\oplus T^*)$ at T^*: this is a skew map $T^* \to T$, i.e. an element $\pi \in C^\infty(\bigwedge^2 T)$ s.t. $[\theta\pi,\theta\pi] = \theta^3\pi^* H \implies [\pi,\pi] = 0$. Thus, $\frac{d}{d\theta}(e^{\theta\mathbb{J}}T^*) = \pi$, and $\pi : \xi \mapsto \pi_T\mathbb{J}\xi$ is a Poisson structure, and we can split

$$\mathbb{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix} \tag{58}$$

The proof of the theorem follows from the following two observations:

1. $\Delta = \mathrm{Im}\ (\pi)$ is the image of a Poisson structure and thus a generalized distribution.
2. The symplectic structure on Δ agrees with π, i.e. for $\xi,\eta \in \Delta^*, \omega^{-1}(\xi,\eta) = \langle \mathbb{J}\xi, \eta\rangle = \pi(\xi,\eta)$.

12.2.3 Weinstein Splitting

Now, assume that the foliation is of locally constant rank near $p \in M$.

Theorem 9 (Weinstein Splitting)**.** *For any $p \in (M,\pi)$ Poisson, there exist coordinates $(q_1,\dots,q_r,p_1,\dots,p_r,y_1,\dots,y_\ell)$ s.t.*

$$\pi = \sum_{i=1}^{r} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^{\ell} \phi(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \tag{59}$$

with $\phi(0) = 0$.

Note.
- When $\ell = 0$, this is the Darboux theorem.
- When the rank at p is locally constant, $\phi = 0$ in a neighborhood of p. (Lie's Theorem)

If the rank is locally constant, then $\mathbb{J}$ induces a complex structure J on $\langle y_1,\cdots,y_{2k}\rangle$ which is integrable since $(\pi x, \pi y) = \pi(x,y)$. Moreover, it is independent of the $\{p_i,q_i\}$, as

$$[\mathbb{J}dp_i, \mathbb{J}dy_j] = \mathbb{J}(d\{p_i,y_j\}) = 0 \tag{60}$$

and similarly for q. This gives us a local coordinate system $\mathbb{R}^{2(n-k)} \times \mathbb{C}^k$.

12.2.4 Examples of type jumping

Given a complex structure $\mathbb{J}_J = \begin{pmatrix} -J & \\ & J^* \end{pmatrix}$ and spaces

$$L = T_{0,1} \oplus T^*_{1,0}, \bigwedge^2 L^* = \bigwedge^2 T_{1,0} \oplus (T_{1,0} \otimes T^*_{0,1}) \oplus \bigwedge^2 T^*_{0,1} \tag{61}$$

we can examine deformations $\epsilon \in \bigwedge^2 L^*$ s.t. $d\epsilon + \frac{1}{2}[\epsilon, \epsilon] = 0$.

Example. For $\epsilon \in \bigwedge^2 T_{1,0}$,

$$\left(\bigwedge^2 T_{1,0} \otimes T^*_{0,1} \right) \oplus \bigwedge^3 T_{1,0} \ni \overline{\partial}\epsilon + \frac{1}{2}[\epsilon, \epsilon] = 0 \implies \overline{\partial}\epsilon = 0, [\epsilon, \epsilon] = 0 \tag{62}$$

i.e. ϵ is a holomorphic Poisson structure.

By construction,

$$\begin{pmatrix} 1 & \overline{\epsilon} \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} L \\ \overline{L} \end{pmatrix} = 1 + \epsilon + \overline{\epsilon} \tag{63}$$

Letting $P = \epsilon + \overline{\epsilon}$, we obtain a transformation $\mathbb{J}_J \mapsto e^p \mathbb{J} e^{-P}$,

$$\begin{pmatrix} 1 & P \\ & 1 \end{pmatrix} \begin{pmatrix} J & \\ & -J^* \end{pmatrix} \begin{pmatrix} 1 & -P \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & P \\ & 1 \end{pmatrix} \begin{pmatrix} J & -JP \\ 0 & -J^* \end{pmatrix} = \begin{pmatrix} J & -JP - PJ^* \\ 0 & -J^* \end{pmatrix} = \begin{pmatrix} PJ & 2Q \\ & -J^* \end{pmatrix} \tag{64}$$

for $Q = i(\overline{\epsilon} - \epsilon)$. Thus, the type is given by $n - \text{rk}Q$.

Example. On $\mathbb{C}P^2$, $\bigwedge^2 T_{1,0} = \mathcal{O}(3)$, and $\epsilon \in H^0(\mathcal{O}(3))$.

12.3 Spinorial Description

Recall that $\mathbb{J}$ determines as is determined by the $+i$-eigenbundle L. Set $pi : L \to T \otimes \mathbb{C}$ to be the map $\pi(L) = E \subset T \otimes \mathbb{C}$. Since $L = L(E, \epsilon), k_L = \langle e^\epsilon \Omega \rangle$, i.e. k_L is generated by products $\phi = e^{B+i\omega}\theta_1 \wedge \cdots \wedge \theta_k$ when $\langle \theta_1, \ldots, \theta_k \rangle = \text{Ann } E$.

Note. However,

1. Let $\xi \in T^*$ be real: then $\xi = \alpha + \overline{\alpha} \in L \oplus \overline{L} \implies \mathbb{J}\xi = i(\alpha + \overline{\alpha})$ and $\pi(\alpha) + \pi(\overline{\alpha}) = 0 \implies \pi(\mathbb{J}\xi) = i\pi(\alpha - \overline{\alpha}) = 2i\pi(\alpha) = -2i\pi(\overline{\alpha})$. Therefore $E \cap \overline{E} = \Delta \otimes \mathbb{C}$, with Ann $\Delta = \langle \Omega \wedge \overline{\Omega} \rangle$, and k is the type of $\mathbb{J}$.

2. $f^*\omega$ is nondegenerate on Δ, as $\langle \phi, \overline{\phi} \rangle \neq 0 \Leftrightarrow \langle e^{B+i\omega}\Omega, e^{B-i\omega}\overline{\Omega} \rangle \neq 0 \Leftrightarrow \langle e^{2i\omega}\Omega, \overline{\Omega} \rangle \neq 0 \Leftrightarrow \omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0$.

Problem. Show that $\omega^{-1} = \pi|_\Delta$.

Given coordinates $(x_1, \ldots, x_{n-k}, p_1, \ldots, p_{n-k}, z_1, \ldots, z_k)$ for $\mathbb{R}^{2(n-k)}_{\omega_0} \times \mathbb{C}^k$, $\omega_0 = \omega|_\Delta$, $\mathbb{J}$ has a general spinor $\phi = e^{B+i\omega} dz_1 \wedge \cdots \wedge dz_k$ around each regular point. Here, we are fixing the splitting so that $H = 0$. Now, $d\phi = \alpha \cdot \phi = (X + \xi) \cdot \phi = d(B + i\omega) \wedge \phi$: by degree considerations, $i_X \Omega = 0$ and $i_X(B + i\omega) + \xi = 0$, so $d\phi = 0$ and $d(B + i\omega) \wedge \Omega = 0$, giving us ∞-integrability.

Theorem 10. *$\phi = e^{B'+i\omega_0}\Omega$ with B' closed, i.e. $\mathbb{J}$ is equivalent to $\mathbb{R}^{2(n-k)}_{\omega_0} \times \mathbb{C}^k$.*

Proof. The general strategy is to transfer to some $e^{B+i\omega}\Omega$ and use the freedom available to make B closed. Using the splitting on $\mathbb{R}^{2(n-k)}_{\omega_0} \times \mathbb{C}^k$, we have a decomposition $d = d_f + \partial + \overline{\partial}$. Set $A = B + i\omega$: then A breaks up into a triangle

$$\begin{pmatrix} A^{200} & & \\ A^{110} & A^{101} & \\ A^{020} & A^{011} & A^{002} \end{pmatrix} \tag{65}$$

which acts effectively via exponentiation on Ω^{0k0}. Note that, via averaging, we have $\omega_0 = \omega|_\Delta = \frac{i}{2}(A^{200} - \overline{A^{200}})$. Our goal is to modify the triangle $(A^{110}, A^{020}, A^{011})$ so that A^{101}, A^{002} enter only in the real part of A. To this end, let C^{011} be any real form, and set

$$\begin{aligned} A' &= A^{200} + (A^{101} + \overline{A^{101}}) + (A^{002} + \overline{A^{002}}) + C^{011} \\ &= \left(\frac{1}{2}(A^{200} + \overline{A^{200}}) + A^{101} + \overline{A^{101}} + A^{002} + \overline{A^{002}} + C^{011}\right) + \frac{1}{2}(A^{200} + \overline{A^{200}}) = B' + i\omega_0 \end{aligned} \tag{66}$$

The condition that $dA \wedge \Omega = 0$ gives four constraints on the A^{ijk}:

$$\begin{aligned} (a) d_f A^{200} &= 0 \\ (b) \overline{\partial} A^{200} + d_f A^{101} &= 0 \\ (c) \overline{\partial} A^{101} + d_f A^{002} & 0 \\ (d) \overline{\partial} A^{002} &= 0 \end{aligned} \tag{67}$$

The desire for B' to be closed requires $(dB')^{012} = (dB')^{111} = 0$, which gives the following two constraints:

$$\begin{aligned} \partial A^{002} + \overline{\partial} C &= 0 \\ \partial A^{101} + d_f C + \overline{\partial A^{101}} &= 0 \end{aligned} \tag{68}$$

We obtain the desired C via the Dolbeault lemma. For the first constraint, note that (d) $\implies A^{002} = \overline{\partial}\alpha^{001}$. Thus

$$\begin{aligned} (1) \Leftrightarrow \overline{\partial} C + \partial\overline{\partial}\alpha = 0 \Leftrightarrow \overline{\partial}(C - \partial\alpha) = 0 \Leftrightarrow \overline{\partial}(C - \partial\alpha - \overline{\partial}\overline{\alpha}) = 0 \\ \Leftrightarrow C - \partial\alpha - \overline{\partial}\overline{\alpha} = \overline{\partial}\psi \Leftrightarrow C = \partial\alpha + \overline{\partial}\overline{\alpha} + i\partial\overline{\partial}\chi \end{aligned} \tag{69}$$

for χ a real function. For the second constraint, note that (c) is true $\Leftrightarrow 0 = \overline{\partial} A^{101} + d_f A^{002} = \overline{\partial}(A^{101} - d_f\alpha) \implies A^{101} = d_f\alpha + \overline{\partial}\beta^{100}$ for β a 100-form. This implies that

$$(2) \Leftrightarrow \partial(d_f\alpha + \overline{\partial}\beta) + \overline{\partial}(d_f\overline{\alpha} + \partial\overline{\beta}) + d_f(\partial\alpha + \overline{\partial}\overline{\alpha} + i\partial\overline{\partial}\chi) = 0 \Leftrightarrow \overline{\partial}\partial(\beta - \overline{\beta}) = \mathrm{i}d_f\partial\overline{\partial}\chi \tag{70}$$

Moreover, (b) is true $\Leftrightarrow \overline{\partial} A^{200} + d_f A^{101} = 0 \Leftrightarrow d_f\overline{\partial}\beta = -\overline{\partial} A^{200}$. Thus, $d_f\overline{\partial}\partial(\beta - \overline{\beta}) = \overline{\partial}\partial(A^{200} - \overline{A^{200}}) = 0$, so we can choose the desired χ. □

Corollary 6. *A GCS on an exact Courant algebroid is locally equivalent, near a regular point, to $\mathbb{R}^{2(n-k)}_{\omega_0} \times \mathbb{C}^k$.*

12.3.1 More Examples of Type Jumping

Recall that we say type jumping via the operator $e^{\beta+\overline{\beta}}\mathbb{J}_J e^{-(\beta+\overline{\beta})}$. We can see this behavior more explicitly using forms. Recall that a complex structure on $\mathbb{C}^2$ a representation by a spinor $\phi = dz_1 \wedge dz_2$. Let $\beta \in H^0(\bigwedge^2 T)$ be a holomorphic section, e.g. $\beta = z_1\partial_1 \wedge \partial_2$ (obviously holomorphic). Then

$$e^{\beta}\phi = e^{\beta+\overline{\beta}}\phi = dz_1 \wedge dz_2 + i_{z_1\partial_1\wedge\partial_2} dz_1 \wedge dz_2 = z_1 + dz_1 \wedge dz_2 \tag{71}$$

At $z_1 = 0$, this gives the complex structure $dz_1 \wedge dz_2$. Outside $z_1 = 0$, we have $z_1(1 + \frac{dz_1+dz_2}{z_1}) \sim e^{B+i\omega}$, where $B + i\omega = \frac{dz_1+dz_2}{z_1}$.

12.3.2 Interpolation

Suppose (g, I, J) is a Hyperk ahler structure, i.e. $(I, g), (J, g)$ are K ahler and $IJ = -JI$. Then $(K = IJ, g)$ is another integrable K ahler structure, and one obtains a family of complex structures $\{aI + bJ + cK | a^2 + b^2 + c^2 = 1\}$ parameterized by S^2, all of which are K ahler w.r.t. g.

Remark. This places a strong constraint on g (reduction of holonomy, Ricci-flat metric, i.e. Einstein) but does not imply that the Riemann curvature is 0. The only known compact examples known are

- $K3$ surface
- Flat T^4
- $\mathrm{Hilb}^n(K3)$
- $\mathrm{Hilb}^n(T^4)$
- Two examples in dimensions 12 and 20 (O'Grady).

Setting $\omega_J I = gJ, \omega_K = gK$, one obtains

$$w_J I = gJI = -gIJ = I^* gJ = I^*\omega_J \tag{72}$$

Moreover, considering the GCSs

$$\mathbb{J}_I = \begin{pmatrix} I & \\ & -I^* \end{pmatrix}, \mathbb{J}_{\omega_J} = \begin{pmatrix} & -\omega_J^{-1} \\ \omega_J & \end{pmatrix}, \mathbb{J}_{\omega_K} = \begin{pmatrix} & -\omega_k^{-1} \\ \omega_k & \end{pmatrix} \tag{73}$$

one obtains the relations

$$\mathbb{J}_I\mathbb{J}_{\omega_J} = \begin{pmatrix} & -I\omega_J^{-1} \\ -I^*\omega_J & \end{pmatrix} = \begin{pmatrix} & -\omega_J^{-1}I^* \\ -\omega_J I & \end{pmatrix} = -\mathbb{J}_{\omega_J}\mathbb{J}_I \tag{74}$$

Similarly, $\mathbb{J}_I\mathbb{J}_{\omega_K} = -\mathbb{J}_{\omega_K}\mathbb{J}_I$ and $\mathbb{J}_{\omega_J}\mathbb{J}_{\omega_K} = -\mathbb{J}_{\omega_K}\mathbb{J}_{\omega_J}$, whereas $\mathbb{J}_I\mathbb{J}_{\omega_I} = \mathbb{J}_{\omega_I}\mathbb{J}_I$. Thus, $(a\mathbb{J}_I + b\mathbb{J}_{\omega_K} + c\mathbb{J}_{\omega_J})^2 = -(a^2 + b^2 + c^2)$, giving a 2-sphere of GCSs interpolating $I \to \omega_J$.

Problem. Show that the intermediate structures are all B-field transforms of symplectic forms.

Note. On $\mathbb{C}P^2$, for the complex case $\mathbb{J}_J$, $K = \Omega^n$, so $K = \emptyset(3)$ and $c_1(K) = -3$. For $\mathbb{J}_\omega$, on the other hand, $K = \langle e^{i\omega}\rangle$ and $c_1(K) = 0$. So we see that we can never interpolate complex to symmetric. In fact, for any even general complex structure,

$$K_{\mathbb{J}} \subset \bigwedge^{ev} T^* \otimes \mathbb{C} = \bigwedge^0 \oplus \bigwedge^2 \oplus \bigwedge^4 \tag{75}$$

there is a canonical projection $s : K_{\mathbb{J}} \to \bigwedge^0 = \mathbb{C}$ (i.e. $s \in C^\infty(K_{\mathbb{J}}^*)$) which vanishes when type jumps off of zero. Hence, we see that for a generic GCS in four dimensions, the type change locus is PD to $c_1(K)$.

Example. In dimension 4, one has types $\{0, 1, 2\}$, so an odd GCS corresponds to a four-manifold foliated by 2-d symplectic leaves and transverse complex structure, e.g. $\Sigma_\omega \times \Sigma_J$ or a symplectec surface bundle over a complex Riemann surface.

Example. In dimension 6, one has types $\{0, 1, 2, 3\}$, and one can construct an odd GCS by deforming the complex structure by a holomorphic Poisson structure (here, the Poisson condition is nontrivial). 0-2 structures?

Problem. Construct an interesting even GCS on a compact 6-manifold.

We now consider examples on Hyperk ahler manifolds. Recall that, for a K ahler manifold one has maps

$$\begin{array}{ccc} T & \xrightarrow{\;g\;} & T^* \\ {\scriptstyle J}\nwarrow & & \nearrow{\scriptstyle \omega} \\ & T & \end{array} \tag{76}$$

s.t. J, ω are integrable, $g = -\omega J$, and $g^* = g \Leftrightarrow J^*\omega = -\omega J$. Thus,

$$\begin{aligned} G = \mathbb{J}_J \mathbb{J}_\omega &= \begin{pmatrix} -J & \\ & J^* \end{pmatrix} \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} = \begin{pmatrix} 0 & J\omega^{-1} \\ J^*\omega & 0 \end{pmatrix} \\ &= \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix} = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} \begin{pmatrix} -J & \\ & J^* \end{pmatrix} = \mathbb{J}_\omega \mathbb{J}_J \end{aligned} \tag{77}$$

is a generalized Riemannian metric. The integrability condition can be rephrased as $\nabla I = 0$ or $\nabla \omega = 0$. As above, for a Hyperk ahler manifold, we have almost complex structures (I, J, K) which are K ahler w.r.t. g and satisfy quaternion relations, thereby giving us a 2-sphere of complex structures $\{aI + bJ + cK\}$. This gives us an integrable complex structure which is K ahler w.r.t. g for $\{(a, b, c) \in S^2\}$.
Now, the relations $\nabla I = 0, \nabla J = 0, \nabla K = 0$ reduce the holonomy of our manifold: the first reduces it $U(n)$, while the second reduces it to the quaternionic unitary group $U(n)_I \cap U(n)_J = \mathrm{Sp}(n)$. This is modeled as follows: set (V, I) to be a complex vector space, with dual V^* and anti-complex space $\overline{V} \cong_{\mathbb{R}}$ with action $i \cdot x = -ix$. Then, in the category of vector spaces with $\mathbb{C}$-linear maps, one has a diagram

$$\begin{array}{ccc} \overline{V} & \xrightarrow{\;h\;} & V^* \\ {\scriptstyle J}\nwarrow & & \nearrow{\scriptstyle Q} \\ & V & \end{array} \tag{78}$$

with Q a complex symplectic form and $h = g + ig(J\cdot, \cdot)$ the induced hermitian metric. Note that J is "anti-linear", in the sense that $Ji = -iJ \implies JI - iIJ$. One thus finds that the holonomy reduction forces the Ricci flow to be trivial, though the whole Riemann tensor need not vanish.
Finally, recall that the only known compact examples are the $K3$ and T^4 surfaces, the Hilbert schemes of both, and the two examples of O'Grady in dimensions 12 and 20. Except for the T^4 and $\mathrm{Hilb}^n(T^4)$, the metrics on these manifolds are not explicit, as they rely on Yau's existence theorem of Ricci flat metrics on K ahler manifolds with holomorphic trivial canonical bundle ($Q \wedge \cdots \wedge Q \neq 0$).

12.3.3 Intermediate Types

As earlier, given a Hyperk ahler structure $(g, I, J, K = IJ)$ and setting $\omega_I = gI, \omega_J = gJ, \omega_K = gK$, we have an S^2-parameterized family of structures $a\mathbb{J}_I + b\mathbb{J}_{\omega_J} + c\mathbb{J}_{\omega_K}$. Moreover, observe that $\mathbb{J}_I \mathbb{J}_{\omega_J} = -\mathbb{J}_{\omega_J} \mathbb{J}_I$, so

$$\mathbb{J} = a\mathbb{J}_I + b\mathbb{J}_{\omega_J} = \begin{pmatrix} -aI & -b\omega_J^{-1} \\ b\omega_J & aI^* \end{pmatrix} \tag{79}$$

is generalized almost-complex for $a^2 + b^2 = 1$. It has Poisson structure $-b\omega_J^{-1} = -\omega^{-1}$, so $\mathbb{J}$ could be a B-field transform

$$\begin{pmatrix} 1 & \\ B & 1 \end{pmatrix}\begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}\begin{pmatrix} 1 & \\ -B & 1 \end{pmatrix} = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix} \tag{80}$$

of $\mathbb{J}_{\frac{1}{b}\omega_J}$. This holds if $b\omega_J^{-1}B = -aI$, i.e. $B = -\frac{a}{b}w_J I = \frac{a}{b}\omega_K$.

Problem. Check that

$$\frac{1}{b}\omega_J + \left(\frac{a}{b}\right)^2 b\omega_K\omega_J^{-1}\omega_K = \frac{1-a^2}{b^2}\omega_J = b\omega_J \tag{81}$$

Thus, we find that $\mathbb{J} = e^{\frac{a}{b}\omega_k}\mathbb{J}_{\frac{1}{b}\omega_J}e^{-\frac{a}{b}\omega_K}$ is integrable.
In another direction, a small deformation of $\mathbb{J}_I$ by a holomorphic Poisson structure is a B-symplectic structure, e.g. take $\beta = (\omega_J + i\omega_K)^{-1}, \overline{\partial}\beta = 0, [\beta, \beta] = 0$.

Problem. Show that $\omega_J + i\omega_K$ is a holomorphic, nondegenerate $(2, 0)$-form and therefore $\beta = (\omega_J + i\omega_K)^{-1}$ is a holomorphic, Poisson, nowhere-vanishing bivector field. Thus, the β-transform is of symplectic type: determine it explicitly.

12.3.4 Generalized K ahler Geometry

Starting with (I, ω_I) in a Hyperk ahler manifold, one can do an infinitesimal deformation by a bivector $t\omega_J^{-1}$ (the real part of the holomorphic Poisson structure $(\omega_J + i\omega_K)^{-1}$). (...)
Thus, the generalized K ahler structure $(\mathbb{J}_A, \mathbb{J}_B)$ induces a $\mathbb{Z}\times\mathbb{Z}$-grading on complex differetial forms

$$S^{\cdot}\otimes\mathbb{C} = \bigoplus_{\substack{p+q\cong n \mod 2 \\ p+q\le n}} \mathcal{U}^{p,q} \tag{82}$$

and that

$$d_H = \delta_+ + \delta_- + \overline{\delta}_- + \overline{\delta}_+ \tag{83}$$

maps $\mathcal{U}^{p,q}$ to $\mathcal{U}^{p+1,q+1}\oplus\mathcal{U}^{p+1,q+1}\oplus\mathcal{U}^{p+1,q-1}\oplus\mathcal{U}^{p-1,q+1}\oplus\mathcal{U}^{p-1,q-1}$. Since $\Delta_{d_H} = \frac{1}{4}\Delta_{\delta_\pm}(-)$, we obtain the Hodge decomposition

$$H_H^*(M, \mathbb{C}) = \bigoplus \mathcal{H}^{p,q} \tag{84}$$

Now, recall that the key observation leading to the K ahler identities was $*|_{\mathcal{U}^{p,q}} = i^{p+q}$

Example. Define the *twisted Betti numbers* to be the values $b^{ev/od} = \dim\, H_H^{ev/od}(M)$, where, if $[H] = 0$, $b^{ev} = \sum_k b^{2k}, b^{od} = \sum_k b^{2k+1}$. Consider the four-dimensional case as given before: then, if the generalized K ahler form is of type (ev, ev), one finds that b^{od} must be even as well, since the action of complex conjugation is reflected through $\mathcal{U}^{0,0}$. Opposingly, if the generalized K ahler form is of type (od, od), b^{ev} must be even. In particular, this implies that on $\mathbb{C}P^2$, there are no (od, od) generalized K ahler structures (since $b^{ev} = 1 + 1 + 1 = 3$).
Now, recall that $* = (i)^{p+q}$ satisfies the identity $\alpha(\alpha(*)\phi) = \star\phi$: in four dimensions, this implies that $\alpha(*) = (-1)^{4*3/2}* = *$ and $\alpha(\phi) = \phi$ is degrees $0, 1, 4$, $-\phi$ in degrees $2, 3$. Applying this to the (ev, ev) case, we find that $\mathcal{U}^{0,0} = (\Omega^0 + \Omega^4)_+ + \Omega^2_-$, while $\mathcal{U}^{-2,0} + \mathcal{U}^{0,2} + \mathcal{U}^{2,0} + \mathcal{U}^{0,-2} = (\Omega^0 + \Omega^4)_- + \Omega^2_+$. Opposingly, in the (od, od) case, we find that $\mathcal{U}^{0,0} = (\Omega^{1,3})_-$, while $\mathcal{U}^{-1,1}\oplus\mathcal{U}^{1,-1} = \Omega^2_- + (\Omega^0 + \Omega^4)_+$ and $\mathcal{U}^{1,1}\oplus\mathcal{U}^{-1,-1} = \Omega^2_+ + (\Omega^0 + \Omega^4)_-$.
Finally, if $[H] = 0$, $*$ induces a splitting on $H^2 = b^2_+ + b^2_-$. Thus, in the (ev, ev) case, b^2_+ is odd and $b_1 = b_3$ is even, while in the (od, od) case, both $b^2_\pm$ are odd, and just b_1 is necessarily even. In particular, for the space $\mathbb{C}P^2\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\#\overline{\mathbb{C}P^2}$, one has twisted Betti numbers $1, 0, 4, 0, 1$.

12.4 Introduction to Hermitian Geometry

Let $G = -\mathbb{J}_A\mathbb{J}_B$: decomposing $E = C_+ \oplus C_-$ into $\pm$-definite spaces, ones finds that $C_\pm = \text{Ker } (G \mp 1)$, i.e. $P_\pm = \frac{1 \pm G}{2}$ are the projection operators to $C_\pm$, so that $P_\pm^2 = P_\pm$. Recall that, given $X \in T$, one has a unique pair of lifts $X^\pm$ to $C_\pm$. We previously obtained $C_\pm = \text{Gr}(b \pm g)$ in an isotropic splitting, so

$$g(X,Y) = \langle X^+, Y^+ \rangle = \langle X^-, Y^- \rangle \tag{85}$$

independent of the isotropy choice. Now, since G commutes with $\mathbb{J}_A$ and $\mathbb{J}_B$, the $C_\pm$ are complex sub-bundles, with $\mathbb{J}_A = \mathbb{J}_B$ on C_+ and $\mathbb{J}_A = -\mathbb{J}_B$ on C_-. Via the isomorphism $\pi : C_\pm \to T$, any structure on $C_\pm$ can be transported to T. In particular, the complex structure on $C_\pm$ gives two almost complex structures J_+, J_- on T, both of which are g-orthogonal (since $\mathbb{J}_A$ preserves $\langle\rangle$ on $C_\pm$). That is, we obtain *almost-Hermitian structures* $(g, J_+), (g, J_-)$ on T.

Proposition 7. *Choose the unique splitting for E where $b = 0$, i.e. $E = (GT^*) \oplus T^* = T \oplus T^*$. Then $(\mathbb{J}_A, \mathbb{J}_B)$ can be reconstructed from (g, J_+, J_-) as follows:*

- *$\mathbb{J}_A$ is J_+ on C_+, J_- on C_-*
- *$\mathbb{J}_B$ is J_- on C_+, J_+ on C_-*

That is,

$$\begin{aligned}
\mathbb{J}_{A/B} &= \pi|_{C_+}^{-1} J_+ \pi P_+ \pm \pi|_{C_-}^{-1} J_- \pi P_- \\
&= \frac{1}{2}\begin{pmatrix} 1 \\ g \end{pmatrix} J_+ \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & g^{-1} \\ g & 1 \end{pmatrix} \pm \frac{1}{2}\begin{pmatrix} 1 \\ -g \end{pmatrix} J_- \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -g^{-1} \\ -g & 1 \end{pmatrix} \\
&= \frac{1}{2}\left(\begin{pmatrix} 1 \\ g \end{pmatrix}\begin{pmatrix} J_+ & J_+ g^{-1} \end{pmatrix} \pm \begin{pmatrix} 1 \\ -g \end{pmatrix}\begin{pmatrix} J_- & -J_- g^{-1} \end{pmatrix}\right)
\end{aligned} \tag{86}$$

Setting $\omega_\pm = gJ_\pm, w_\pm^{-1} = -J_\pm g^{-1}$, one obtains

$$\begin{aligned}
\mathbb{J}_{A/B} &= \frac{1}{2}\left(\begin{pmatrix} J_+ & -\omega_+^{-1} \\ \omega_+ & -J_+^* \end{pmatrix} \pm \begin{pmatrix} J_- & \omega_-^{-1} \\ -\omega_- & -J_-^* \end{pmatrix}\right) \\
&= \frac{1}{2}\begin{pmatrix} J_+ \pm J_- & -\omega_+^{-1} \pm \omega_-^{-1} \\ \omega_+ \mp \omega_- & -J_+^* \mp J_-^* \end{pmatrix}
\end{aligned} \tag{87}$$

12.4.1 Condition on Types

The above expression implies that $\pi_{A/B} = \omega_+^{-1} \mp \omega_-^{-1}$ are real Poisson structures and $\omega_+^{-1} = -J_+ g^{-1}$, with types

$$\begin{aligned}
\text{type}(\mathbb{J}_A) &= \frac{1}{2}\dim_{\mathbb{R}}(\text{Ker } \pi_A = \text{Ker } (J_+ - J_-)) \\
\text{type}(\mathbb{J}_B) &= \frac{1}{2}\dim_{\mathbb{R}}(\text{Ker } \pi_B = \text{Ker } (J_+ + J_-))
\end{aligned} \tag{88}$$

Note that

$$(\star)[J_+, J_-] = (J_+ + J_-)(J_+ - J_-) \tag{89}$$

Thus,

1. $(J_+ - J_-), (J_+ + J_-)$ have linearly independent kernels.

2. $\star \implies \text{Ker } (J_+ - J_-) \oplus \text{Ker } (J_+ + J_-) \subset \text{Ker } [J_+, J_-]$

3. If $[J_+, J_-]x = 0$, then

$$x = \frac{x + J_+J_-x}{2} + \frac{x - J_+J_-x}{2} \tag{90}$$

and $(J_+ + J_-)(x + J_+J_-x) = 0$. Thus, $\text{Ker } (J_+ + J_-) \oplus \text{Ker } (J_+ - J_-) = \text{Ker } [J_+, J_-]$, and $\text{type}(\mathbb{J}_A) + \text{type}(\mathbb{J}_B) = \frac{1}{2}\dim_{\mathbb{R}}\text{Ker } [J_+, J_-]$.

Corollary 7. $\text{type}(A) + \text{type}(B) \leq n$ *on* M^{2n}.

It immediatly follows from this that, since $\text{type}(A) + \text{type}(B) = n$ everywhere $\Leftrightarrow [J_+, J_-] = 0$, then the pair $(\text{type}(A), \text{type}(B))$ is constant on a connected manifold.

12.4.2 Integrability

As above, we have a map with structure actions $\mathbb{J}_A \circlearrowleft C_\pm \to T \circlearrowleft J_\pm$ from our decomposed bundle to T. Note that the complexifications of these bundles are given by

$$C_+ \otimes \mathbb{C} = L_+ \oplus \overline{L}_+, C_- \otimes \mathbb{C} = L_- \oplus \overline{L}_- \tag{91}$$

, where $L_+ = L_A \cap L_B, L_- = L_A \cap \overline{L}_B$. Now, L_A, L_B are integrable $\implies L_\pm$ are Courant integrable $\implies \pi(L_\pm) = T^{1,0}_\pm$ are Lie integrable $\implies J_\pm$ are integrable $\implies (J_\pm, g)$ are both Hermitian. With the chosen splitting, we have

$$L_+ = \{X + gX : X \in T^{1,0}_+\} = \{X - i\omega_+X : X \in T^{1,0}_+\} \tag{92}$$

L_+ is closed under H-Courant $\Leftrightarrow$

$$\forall X, Y \in T^{1,0}_+, i_Xi_Y(H - id\omega_+) = 0 \tag{93}$$

Similarly,

$$L_- = \{X - gX : X \in T^{1,0}_-\} = \{X + i\omega_-X : X \in T^{1,0}_-\} \tag{94}$$

and L_- is closed under H-Courant $\Leftrightarrow$

$$\forall X, Y \in T^{1,0}_-, i_Xi_Y(H + id\omega_-) = 0 \tag{95}$$

We can rewrite this as

$$\begin{aligned} i_Xi_Y(H \mp id\omega_\pm) &= 0 \\ i_Xi_Y(H \mp i(\partial\overline{\partial})\omega_\pm) &= 0 (\text{since } i_Xi_Y\overline{\partial}\omega_\pm = 0) \\ i_Xi_Y(H \pm d^c_\pm\omega_\pm) &= 0 \\ H \pm d^c_\pm\omega_\pm &= 0 \end{aligned} \tag{96}$$

That is, for a generalized K ahler manifold, we must have $H = d^c_+\omega_+ = -d^c_-\omega_-$ in order that $J_\pm$ is integrable.

Theorem 11. *An abstracted defined* $\mathbb{J}_{A/B}$ *on* $T \oplus T^*, H$ *defines a generalized K aher structure* $\Leftrightarrow H = d^c_+\omega_+ = -d^c_-\omega_-$. *That is, a generalized K ahler structure over a b-field is a triple* (g, J_+, J_-) *s.t.* $d^c_+\omega_+ = -d^c_-\omega_-$.