13 Lecture 18 (Notes: K. Venkatram)

13.1 Generalized K ahler Geometry

Let $(\mathbb{J}_A, \mathbb{J}_B)$ be a generalized K ahler structure: then $G = -\mathbb{J}_A\mathbb{J}_B$ is a generalized metric, and taking the decomposition $T \oplus T^* = C_+ \oplus C_-, C_\pm = \Gamma_{\pm g}$ gives $\mathbb{J}_A|_{C_+} = \mathbb{J}_B|_{C_+}, \mathbb{J}_A|_{C_-} = -\mathbb{J}_B|_{C_-}$. Thus, we obtain two complex structures J_+, J_- on T by transport, i.e. $J_+X = \pi\mathbb{J}_A X^+$ and $J_-X = \pi\mathbb{J}_A X^-$. Since $\mathbb{J}_A$ is compatible with G, this implies that $(J_+, g), (J_-, g)$ are almost Hermitian. Further, given the splitting of the Courant algebroid, $\mathbb{J}_A, \mathbb{J}_B$ can be reconstructed from (g, J_+, J_-) by

$$\begin{aligned} \mathbb{J}_A &= J_+|_{C_+} + J_-|_{C_-} \\ \mathbb{J}_B &= J_+|_{C_+} - J_-|_{C_-} \end{aligned} \tag{97}$$

thus giving the formula

$$\mathbb{J}_{A/B} = \frac{1}{2}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^* \pm J_-^*) \end{pmatrix} \tag{98}$$

13.1.1 Integrability

As shown earlier, the integrability of $(\mathbb{J}_A, \mathbb{J}_B)$ is equivalent to the Courant involutivity of L_A, L_B. Specifically, note that

$$\begin{aligned} (T \oplus T^*) \otimes \mathbb{C} = L_A \oplus \overline{L}_A = L_B \oplus \overline{L}_B &= (L_A \cap L_B) \oplus (L_A \cap \overline{L}_B) \oplus (\overline{L}_A \cap L_B) \oplus (\overline{L}_A \cap \overline{L}_B) \\ &= L_+ \oplus L_- \oplus \overline{L}_- \oplus \overline{L}_+ \end{aligned} \tag{99}$$

Thus, the complex structures on $C_\pm$, and thus on T, are described by the decompositions $C_+ \otimes \mathbb{C} = L_+ \oplus \overline{L}_+, C_- \otimes \mathbb{C} = L_- \oplus \overline{L}_-$, and the dimensions of the four spaces on the rhs are the same. Finally, since $T^+_{1,0} = +i$ for $J_+ = L_+$ (and similarly, $T^-_{1,0} = L_-$), we have integrability $\Leftrightarrow L_A, L_B$ are involutive $\implies L_\pm$ is involutive. The latter impliciation is in fact an iff:

Proposition 8. *$L_\pm$ involutive $\implies L_+ \oplus L_-, L_+ \oplus \overline{L_-}$ involutive.*

Proof. Using the fact that

$$\langle [a,b], c\rangle \cdot \phi = [[[d_H, a], b], c] \cdot \phi = a \cdot b \cdot c \cdot d_H \phi \tag{100}$$

for any ϕ pure, $a, b, c \in L_\phi$, we find that $\langle [a,b], c\rangle$ defined a tensor in $\bigwedge L_\phi^*$. Let $a \in L_+, b \in L_-$ be elements. Then, for any $x \in L_+$, $\langle [a,b], x\rangle = \langle [x,a], b\rangle = 0$. Similarly, for any $x \in L_-$, $\langle [a,b], x\rangle = \langle [b,x], a\rangle = 0$. Thus, $[a,b] \in L_+ \oplus L_-$. □

However, as we saw last time,

$$L_\pm = \{X \pm gX | X \in T_\pm^{1,0}\} = \{X \mp i\omega_\pm X | X \in T_\pm^{1,0}\} \tag{101}$$

and so $L_\pm$ are integrable $\Leftrightarrow T_\pm^{1,0}$ are integrable and $i_X i_Y (H \mp i d\omega_\pm) = 0 \forall X, Y \in T_\pm^{1,0}$. Using the integrability of $J_\pm$, we can write the latter expression as $i_X i_Y (H \mp i(\partial_\pm + \overline{\partial}_\pm)\omega_\pm) = 0 \forall X, Y \in T_\pm^{1,0}$. Since $\overline{\partial}_\pm \omega_\pm$ is of type $1,2$, it is killed, and

$$i_X I_Y (H \pm d^c_\pm \omega_\pm) = 0 \Leftrightarrow H \pm d^c_\pm \omega_\pm = 0 \Leftrightarrow \begin{cases} d^c_+\omega_+ + d^c_-\omega_- = 0 \\ d^c_+\omega_+ = -H \end{cases} \tag{102}$$

Finally, we obtain the following result.

Theorem 12. *Generalized K ahler structures on the exact Courant algebroid $E \to M$, modulo non-closed B-field transforms (choice of splitting) are equivalent to bi-Hermitian structures (g, J_+, J_-) s.t. $d^c_+\omega_+ + d^c_-\omega_- = 0, dd^c_+\omega_+ = 0$, and $[d^c_+\omega_+] = [E] \in H^3(M, \mathbb{R})$.*

Remark. This geometry was first described by Gates, Hull, Roček as the most general geomtry on the target of a 2-dimensional sigma model constrained to have $N = (2, 2)$ supersymmetry. Note that the special identities giving a (p, q) decomposition of $H^*_H(M, \mathbb{C})$ are a consequence of the special identities required by SUSY. However, they are only clear when viewed in terms of $(\mathbb{J}_A, \mathbb{J}_B)$ rather than $J_\pm$.

We can use this theorem to construct several new examples of generalized K ahler and generalized complex structures.

Example. Let G be an even-dimensional, compact, semisimple group, and choose an even-dimensional Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} \otimes \mathbb{C}$. The root system splits into $\pm re$ roots, giving a decomposition $\mathfrak{g} \otimes \mathbb{C} = \tau \oplus \overline{\tau}$ which is closed onder the Lie bracket. Thus, by left or right translating, we get an integrable complex structure on G, and since the root spaces are killing-orthogonal, we have a bi-Hermitian structure (g, J_L, J_R), with g the killing form. Now, recall the Cartan 3-form $H(X, Y, Z) = g([X, Y], Z)$ and notice that

$$\begin{aligned} A = d^c_L\omega_L(X, Y, Z) &= d\omega_L(J_LX, J_LY, J_LZ) = -\omega_L([J_LX, J_LY], J_LZ) + \text{c.p.} \\ &= -g(J_L[J_LX, Y] + J_L[X, J_LY] + [X, Y], Z) + \text{c.p.} \\ &= (2g([J_LX, J_LY], Z) + \text{c.p.}) - 3H(X, Y, Z) = -2A - 3H \end{aligned} \tag{103}$$

Thus, $d^c_L\omega_L = -H$; since the right Lie algebra is anti-isomorphic to the left, $d^c_R\omega_R = H$, and (G, g, J_L, J_R) is a generalized K ahler structure unique w.r.t. H_{cartan}. Finally, we obtain the generalized complex structures

$$\mathbb{J}_{A/B} = \begin{pmatrix} J_L \pm J_R & -(\omega_L^{-1} \mp \omega_R^{-1}) \\ \omega_L \mp \omega_R & -(J_L^* \pm J_R^*) \end{pmatrix} \tag{104}$$

on G.
What are their types? Since $\omega_L = gJ_L, \omega_R = gJ_R$,

$$\begin{aligned} -(\omega_L^{-1} \mp \omega_R^{-1}) &= (J_L \mp J_R)g^{-1} \\ J_L \pm J_R &= R_{g*}(R_{g^{-1}*}L_{g*}J \pm JR_{g^{-1}*}L_{g*})L_{g^{-1}*} \end{aligned} \tag{105}$$

Thus, the rank of $(\mathbb{J}_A, \mathbb{J}_B)$ at g is simply $(\text{rk}[J, \text{Ad }_g], \text{rk}\{J, \text{Ad }_g\})$.

Problem. Describe the symplectic leaves of $(\mathbb{J}_A, \mathbb{J}_B)$ for $G = SU(3)$.

In the simplest case, $Q = [J_+, J_-]g^{-1} = 0$, so that type $A +$ type $B = n \implies$ constant types. As earlier, since $[J_+, J_-] = 0$, we have a decomposition $T \otimes \mathbb{C} = A \oplus B \oplus \overline{A} \oplus \overline{B}$, with $A = T^+_{1,0} \cap T^-_{1,0}, B = T^+_{1,0} \cap T^-_{0,1}$. Note that A, B are integrable since $T^+_{1,0}, T^-_{1,0}$ are. Also, note that $A \oplus \overline{A} = \text{Ker } (J_+ - J_-) = \text{Im } (J_+ + J_-) = \text{Im } \pi_A$ is integrable, as is $B \oplus \overline{B}$.

Proposition 9. *A, B are holomorphic subbundles of $T^+_{1,0}$.*

Proof. Define $\overline{\partial}_{X^{0,1}}Z^{1,0} = [X, Z]^{1,0}$. For $Z \in C^\infty(A), X = X_{\overline{A}} + X_{\overline{B}}, [X, Z]^{1,0} = [X, Z]^A + [X, Z]^B$, with the latter term being zero since $[X_{\overline{A}}, Z]$ is still in $A \oplus \overline{A}$ and $[X_{\overline{B}}, Z]$ is in the integrable space $A \oplus \overline{B}$. Thus, A (and similarly B) give $J_\pm$ holomorphic splittings of TM. ∎