14 Lecture 19 (Notes: K. Venkatram)

14.1 Generalized K ahler Geometry

Recall from earlier that a K ahler structure is a pair $\mathbb{J}_J = \begin{pmatrix} J & \\ & -J^* \end{pmatrix}, \mathbb{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$ s.t. $\mathbb{J}_J \mathbb{J}_\omega = \mathbb{J}_\omega \mathbb{J}_J = -\begin{pmatrix} & g^{-1} \\ g & \end{pmatrix} = -G$.

Definition 22. *A* generalized K ahler structure *is a pair* $(\mathbb{J}_A, \mathbb{J}_B)$ *of generalized complex structures s.t.* $-\mathbb{J}_A \mathbb{J}_B = G$ *is a generalized Riemannian metric.*

The usual example has type $(0, n)$ for $\mathbb{J}_A, \mathbb{J}_B$. In fact, as we will show later type $\mathbb{J}_A +$ type $\mathbb{J}_B \le n$ and $\equiv n$ mod 2.

Example.

1. Can certainly apply B-field $(e^B \mathbb{J}_A e^{-B}, e^B \mathbb{J}_B e^{-B})$ and obtain the generalized metric $e^B G e^{-B}$.
2. Going back to hyperk ahler structures, recall that

$$(\omega_J + i\omega_K)I = g(J + iK)I = -gI(J + iK) = I^*(\omega_J + i\omega_K) \tag{106}$$

so $\frac{1}{2}(\omega_J + i\omega_k) = \sigma$ is a holomorphic $(2,0)$-form with $\sigma^n \neq 0$. Note that $\beta = \frac{1}{2}(\omega_J^{-1} - i\omega_k^{-1})$ satisfies $\beta\sigma = \frac{1}{2}(1 - iI) = P_{1,0}$, i.e. it is the projection to the $(1,0)$-form $\beta|_{T^*_{1,0}} = \sigma^{-1}|_{T^{1,0}}$.

Recall that, for β a holomorphic $(2,0)$-bivector field s.t. $[\beta, \beta] = 0$, $e^{\beta + \overline{\beta}} \mathbb{J}_I e^{-\beta - \overline{\beta}}$ is a generalized complex structure. Thus, we have

$$\begin{aligned}
\begin{pmatrix} 1 & t\omega_J^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} I & \\ & -I^* \end{pmatrix} \begin{pmatrix} 1 & -t\omega_J^{-1} \\ & 1 \end{pmatrix} &= \begin{pmatrix} I & -t\omega_j^{-1} I^* \\ & -I^* \end{pmatrix} \begin{pmatrix} 1 & -t\omega_J^{-1} \\ & 1 \end{pmatrix} = \begin{pmatrix} I & -tI\omega_J^{-1} - t\omega_J^{-1} I^* \\ 0 & -I^* \end{pmatrix} \\
&= \begin{pmatrix} I & 2tKg^{-1} \\ & -I^* \end{pmatrix} = \begin{pmatrix} I & -2t\omega_K^{-1} \\ & -I^* \end{pmatrix}
\end{aligned} \tag{107}$$

Now, note that

$$\begin{aligned}
\begin{pmatrix} 1 & t\omega_J^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} & -\omega_I^{-1} \\ \omega_I & \end{pmatrix} \begin{pmatrix} 1 & -t\omega_J^{-1} \\ & 1 \end{pmatrix} &= \begin{pmatrix} t\omega_J^{-1}\omega_I & -\omega_I^{-1} \\ \omega_I & \end{pmatrix} \begin{pmatrix} 1 & -t\omega_J^{-1} \\ & 1 \end{pmatrix} \\
&= \begin{pmatrix} t\omega_J^{-1}\omega_I & -\omega_I^{-1} - t^2\omega_J^{-1}\omega_I\omega_J^{-1} \\ \omega_I & -t\omega_I\omega_J^{-1} \end{pmatrix} \\
&= \begin{pmatrix} tK & (-1 + t^2)\omega_I^{-1} \\ \omega_I & -tK^* \end{pmatrix} \\
&= \sqrt{1 - t^2}\, \mathbb{J}_{\frac{1}{\sqrt{1-t^2}}\omega_I} + t\mathbb{J}_K
\end{aligned} \tag{108}$$

By a previous calculation, this is integrable, and $\mathbb{J}_A = \begin{pmatrix} I & -2t\omega_K^{-1} \\ & -I^* \end{pmatrix}, \mathbb{J}_B = \begin{pmatrix} tK & (-1 + t^2)\omega_I^{-1} \\ \omega_I & -tK^* \end{pmatrix}$ is a generalized K ahler structure of type $(0, 0)$.

Problem. Let (J, ω) be a K ahler structure, β a holomorphic Poisson structure. For $Q = \beta + \overline{\beta}$, when is $e^{tQ} \mathbb{J}_\omega e^{-tQ}$ integrable for small t?

What is the analog of the Hodge decomposition $H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$ for generalized K ahler manifolds. The key element of this decomposition in the case of ordinary K ahler structures is to show that $\Delta_d = \partial\Delta_\partial = \partial\Delta_{\overline{\partial}}$, where $\Delta_d = dd^* + d^*d = (d+d^*)^2$, and d^* is the adjoint of d in an appropriate metric define on forms. The equality of the above decomposition then follows from Hodge theory (that every cohomology class has a unique harmonic representative).

14.2 Hodge Theory on Generalized K ahler Manifolds

Recall the Born-Infeld volume: letting (a_i) be an orthonormal basis for C_+ in $\mathrm{Pin}(T\oplus T^*)$, we have an associated element $-G \in O(n,n)$; letting $\star\psi = \alpha(\alpha(*)\psi)$ denote the generalized Hodge star and $\langle *\phi, \psi\rangle \in \det T^*$ the symmetric volume form, the *Born-Infeld* inner product on $S\otimes\mathbb{C} = \Omega^*(M,\mathbb{C})$ is

$$(\phi,\psi) = \int_M \langle *\phi, \overline{\psi}\rangle \tag{109}$$

This is a Hermitian inner product. Recall also that, if we split $T\oplus T^*$ and $G = \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix}$, then $\langle *\phi,\psi\rangle = \phi\wedge\star\psi = (\phi,\psi)\mathrm{vol}_g$ via the usual Hodge inner product. What is the adjoint of d_H?

Lemma 7. $\langle d\phi,\psi\rangle = (-1)^{\dim\ M}\langle\phi,\partial\psi\rangle$.

Proof. First, note that $\alpha(\phi^{(k)}) = (-1)^{\frac{1}{2}k(k-1)}\phi^{(k)}$. then

$$\begin{aligned} d(\phi\wedge\alpha(\psi)) &= d\phi\wedge\alpha(\psi) + (-1)^k\phi\wedge d\alpha(\psi) \\ d(\alpha(\psi^p)) &= (-1)^{\frac{1}{2}p(p-1)}d\psi^p = (-1)^{\frac{1}{2}p(p-1)+\frac{1}{2}p(p+1)}\alpha(d\psi^p) = -\alpha(d\psi^p) \end{aligned} \tag{110}$$

Thus, $d(\phi\wedge\alpha(\psi)) = \langle d\phi,\psi\rangle + (-1)^n\langle\phi,d\psi\rangle$. □

Lemma 8. *We have the same for* $H\wedge\cdot$.

Corollary 8. *On an even-dimensional manifold,* $\int_M\langle d_H\phi,\psi\rangle = \int_M\langle\phi,d_H\psi\rangle$.

Now

$$h(d_H\phi,\psi) = \int\langle *d_H\phi,\psi\rangle = \int\langle d_H\phi,\sigma(ast)\psi\rangle = \int\langle\phi,d_H\sigma(*)\psi\rangle = \int\langle *\phi, *d_H\sigma(*)\psi\rangle \tag{111}$$

so $d_H^* = *d_H*^{-1}$. As in the classical case, $d_H + d_H^*$ is elliptic, as is $D^2 = \Delta_{d_H}$. By Hodge theory, every twisted deRham cohomology class has a unique harmonic representative.

To perform Hodge decomposition on generalized K ahler manifolds, note that we have two commuting actions on spinors. For $\mathbb{J}_A$, we have the maps $\partial_A : \mathcal{U}_k \to \mathcal{U}_{k+1}$ and $\overline{\partial}_A : \mathcal{U}_k \to \mathcal{U}_{k-1}$, with the associated differential $d_H = \partial_A + \overline{\partial}_A$. Each $\mathcal{U}_k$ must decompose as eigenspaces for $\mathbb{J}_B$, i.e. we can obtain a set of spaces $\mathcal{U}_{r,s}$ which has the pair of eigenvalues (ir, is) for $(\mathbb{J}_A,\mathbb{J}_B)$. Between these spaces, we have horizontal maps given by $L_A, \overline{L_A}$ and vertical maps given by $L_B, \overline{L_B}$, with the associated decompositions

$$\begin{aligned} (T\oplus T^*)\otimes\mathbb{C} &= L_A\cap L_B \oplus L_A\cap\overline{L_B} \oplus \cap L_A\cap L_B \oplus \overline{L_A}\cap\overline{L_B} \\ d_H &= \delta_+ + \delta_- + \overline{\delta_+} + \overline{\delta_-} \end{aligned} \tag{112}$$

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Proposition 10. $\delta_+^* = -\overline{\delta}_+$ *and* $\delta_-^* = \overline{\delta}_-$.

Proof. The identity $\mathbb{J}_A \mathbb{J}_B = -G$ corresponds to the spin decomposition $e^{\frac{\pi}{2}\mathbb{J}_A} \times e^{\frac{\pi}{2}\mathbb{J}_B} = *$. Thus, for $\phi \in \mathcal{U}^{p,q}$, $*\phi = e^{\frac{\pi}{2}\mathbb{J}_A} \times e^{\frac{\pi}{2}\mathbb{J}_B}\phi = i^{p+q}\phi$ and

$$\delta_+^* = (*d_H *^{-1} \phi) = (i^{p+q-2}\overline{\delta}_+ i^{-p-q}\phi) = -\overline{\delta}_+ \tag{113}$$

The other identity follows similarly. □

Corollary 9. *If $\phi \in \mathcal{U}^{p,q}$ is closed (i.e. $d_H\phi = 0$) then it is Δ closed as well.*

By our above decomposition of d_H and the implied decomposition of d_H^*, we find that $\frac{1}{2}(d_H + d_H^*) = \delta_- + \delta_-^*$ and $\frac{1}{2}(d_H - d_H^*) = \delta_+ + \delta_+^*$, so that $\frac{1}{4}\Delta_{d_H} = \Delta_{\delta_-} = \Delta_{\delta_+}$. This finally gives us our desired decomposition

$$H_H^*(M, \mathbb{C}) = \bigoplus_{|p+q| \leq n, p+q \equiv n \mod 2} \mathcal{H}^{p,q}_{\Delta_{d_H}} \tag{114}$$