# 2 Lecture 2 (Notes: A. Rita)

## 2.1 Comments on previous lecture

(0) The Poincaré lemma implies that the sequence

$$\dots \longrightarrow C^\infty(\wedge^{k-1}T^*) \xrightarrow{d} C^\infty(\wedge^k T^*) \xrightarrow{d} C^\infty(\wedge^{k+1}T^*) \longrightarrow \dots$$

is an exact sequence of sheaves, even though it is not an exact sequence of vector spaces.

(1) We defined the Lie derivative of a vector field $X$ to be $L_X = [\iota_X, d]$. Since $\iota_X \in \mathrm{Der}^{-1}(\Omega^\cdot(M))$ and $d \in \mathrm{Der}^{+1}(\Omega^\cdot(M))$, we have

$$[\iota_X, d] = \iota_X d - (-1)^{(1)\cdot(-1)} d\iota_X = \iota_X d + d\iota_X$$

(2) $\omega : V \longrightarrow V^*$, $\omega^* = -\omega$ If $\omega$ is an isomorphism, then for any $X \in V$ we have $\omega(X, X) = 0$, so that

$$X \in X^\omega = \mathrm{Ker}\ \omega(X) = \omega^{-1}\mathrm{Ann}\ X$$

Thus, we have an isomorphism $\omega^* : X^\omega / \langle X \rangle \xrightarrow{\cong} \mathrm{Ann}\ X / \langle \omega X \rangle$ and

$$\frac{\mathrm{Ann}\ X}{\langle \omega X \rangle} = \frac{\langle X \rangle^*}{(X^\omega)^*} = \left(\frac{X^\omega}{\langle X \rangle}\right)^*$$

Then using induction, we can prove that $V$ must be even dimensional.

## 2.2 Symplectic Manifolds

(continues the previous lecture)

For a manifold $M$, consider its cotangent bundle $T^*M$ equipped with the 2-form $\omega = d\theta$, where $\theta \in \Omega^1(T^*M)$is such that $\theta_\alpha(v) = \alpha(\pi_*(v))$. In coordinates $(x^1, \dots, x^n, a_1, \dots, a_n)$, we have $\theta = \sum_i a_i dx^i$ and therefore $d\theta = \sum_i da_i \wedge dx^i$, as in the Darboux theorem. Thus, $T^*M$ is symplectic.

**Definition 6.** *A subspace $W$ of a symplectic $2n-$dimensional vector space $(V, \omega)$ is called* isotropic *if* $\omega|_W = 0$.

*$W$ is called* coisotropic *if its $\omega$-perpendicular subspace $W^\omega$ is isotropic.*

*$W$ is called* Lagrangian *if it is both isotropic and coisotropic.*

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There exist isotropic subspaces of any dimension $0, 1, \ldots, n$, and coisotropic subspaces of any dimension $n, n+1, \ldots, 2n$. Hence, Lagragian subspaces must be of dimension $n$.

We have analogous definitions for submanifolds of a symplectic manifold $(M, \omega)$:

**Definition 7.** *$L \overset{f}{\hookrightarrow} (M, \omega)$ is called* isotropic *if $f^*\omega = 0$. When $dim(L) = n$ it is called* Lagrangian.

The graph of $0 \in C^\infty(M, T^*M)$, which is the zero section of $T^*M$, is Lagrangian.

More generally, $\Gamma_\xi$, the graph of $\xi \in C^\infty(M, T^*M)$ is a Lagrangian submanifold of $T^*M$ if and only if $d\xi = 0$. It is in this sense that we say that Lagrangian submanifolds of $T^*M$ are like generalized functions: $f \in C^\infty(M)$ gives rise to $df$, which is a closed 1−form, so $\Gamma_{df} \subset T^*M$ is Lagrangian.

**Proposition 1.** *Suppose we have a diffeomorphism between two symplectic manifolds, $\varphi : (M_0, \omega_0) \to (M_1, \omega_1)$ and let $\pi_i : M_0 \times M_1 \to M_i$, $i = 0, 1$ be the projection maps.*

*Then, Graph$(\varphi) \subset (M_0 \times M_1, \pi_0^*\omega_0 - \pi_1^*\omega_1)$ is Lagrangian if and only if $\varphi$ is a symplectomorphism.*

## 2.3 Poisson geometry

**Definition 8.** *A* Poisson structure *on a manifold $M$ is a section $\pi \in C^\infty(\wedge^2(TM))$ such that $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Shouten bracket.*

**Remark.** $[\pi, \pi] \in C^\infty(\wedge^3(TM))$, so for a surface $\Sigma^{(2)}$, all $\pi \in C^\infty(\wedge^2(TM))$ are Poisson.

This defines a bracket on functions, called the Poisson bracket:

**Definition 9.** *The* Poisson bracket *of two functions $f, g \in C^\infty(\wedge^0(TM))$ is*

$$\{f, g\} = \pi(df, dg) = \iota(df \wedge dg) = [[\pi, f], g]$$

**Proposition 2.** *The triple $(C^\infty(M), \cdot, \{\ ,\})$ is a Poisson algebra, i.e., it satisfies the properties below. For $f, g, h \in C^\infty(\wedge^0(TM))$,*

- *Leibniz rule $\{f, gh\} = \{f, g\}h + g\{f, h\}$*
- *Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$*

**Problem.** Write $\{f, g\}$ in coordinates for $\pi = \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$.

A basic example of a Poisson structure is given by $\omega^{-1}$, where $\omega$ is a symplectic form on $M$, since

$$\left[\omega^{-1}, \omega^{-1}\right] = 0 \Leftrightarrow d\omega = 0 \tag{6}$$

**Problem.** Prove (6) by testing $d\omega(X_f, X_g, X_h)$, for $f, g, h \in C^\infty(M)$.

Poisson manifolds are of interest in physics: given a function $H \in C^\infty(M)$ on a Poisson manifold $(M, \pi)$, we get a unique vector field $X_H = \pi(dH)$ and its flow $Fl^t_{X_H}$. $H$ is called Hamiltonian, and we usually think about it as energy.

We have $X_H(H) = \pi(dH, dH) = 0$, so $H$ is preserved by the flow. What other functions $f \in C^\infty(M)$ are preserved by the flow? A function $f \in C^\infty(M)$ is conserved by the flow if and only if $X_H(f) = 0$, equivalently $\{H, f\} = 0$, $f$ commutes with the Hamiltonian.

If we can find $k$ conserved quantities, $H_0 = H, H_1, H_2, \ldots, H_k$ such that $\{H_0, H_i\} = 0$, then the system must remain on a level surface $Z = \{x : (H_0, \ldots, H_k) = \vec{c}\}$ for all time. Moreover, if $\{H_i, H_j\}$ for all $i, j$ then we get commutative flows $Fl^t_{X_{H_i}}$. If $Z$ is compact, connected, and $k = n$, then $Z$ is a torus $\mathbb{T}^n$, and the trajectory is a straight line in these coordinates. Also, $\mathbb{T}^n$ is Lagrangian.

**Problem.** Describe the Hamiltonian flow on $T^*M$ for $H = \pi^* f$, with $f \in C^\infty(M)$ and $\pi : T^*M \to M$. Show that a coordinate patch for $M$ gives a natural system of $n$ commuting Hamiltonians.

Let us now think about a Poisson structure, $\pi : T^* \to T$ and consider $\Delta = \text{Im}\pi$. $\Delta$ is spanned at each point $x$ by $\pi(df) = X_f$, Hamiltonian vector fields. The Poisson tensor is always preserved:

$$L_{X_f}\pi = [\pi, X_f] = [\pi, [\pi, f]] = [[\pi, \pi], f] + (-1)^{1\cdot 1} [\pi, [\pi, f]] = -[\pi, [\pi, f]]$$

$$\Longrightarrow L_{X_f}\pi = 0$$

If $\Delta_{x_0} = \langle X_{f_1}, \ldots, X_{f_k}\rangle$, then $Fl^{t_1}_{X_1} \circ \ldots \circ Fl^{t_k}_{X_k}(x_0)$ sweeps out $S \ni x_0$ submanifold of $M$ such that $TS = \Delta$.

**Example** (of a generalized Poisson structure)**.** Let $M = \mathfrak{g}^*$, for $\mathfrak{g}$ a Lie algebra, $[\cdot,\cdot] \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{g}$. Then $TM = M \times \mathfrak{g}^*$ and $T^*M = M \times \mathfrak{g}$, and also $\wedge^2(TM) = M \times \wedge^2\mathfrak{g}$, so $[\cdot,\cdot] \in C^\infty(\wedge^2 T\mathfrak{g}^*)$.

Given $f_1, f_2 \in C^\infty(M)$, their Poisson bracket is given by $\{f_1, f_2\}(x) = \langle [df_1, df_2], x\rangle$.

For $f, g \in \mathfrak{g}$ linear functions on M, we have

$$X_f(g) = \langle [f,g], x\rangle = \langle \text{ad}_f g, x\rangle = \langle g, -\text{ad}^*_f x\rangle$$

Thus $X_f = -\text{ad}^*_f$, so the the leaves of $\Delta = \text{Im}\pi$ are coadjoint orbits. If $S$ is a leaf, then

$$0 \longrightarrow N^*_S \longrightarrow T^*|_S \xrightarrow{\pi} T|_S \longrightarrow 0$$

is a short exact sequence and we have an isomorphism $\pi_* : T^*S = \frac{T^*|_S}{N^*S} \xrightarrow{\cong} TS$, which implies that the leaf $S$ is symplectic.

Given $f, g \in C^\infty(S)$, we can extend them to $\tilde{f}, \tilde{g} \in C^\infty(M)$. The Poisson bracket $\left\{\tilde{f}, \tilde{g}\right\}_\pi$ is independent of choice of $\tilde{f}, \tilde{g}$, so $\{f, g\}_{\pi_*} = \left\{\tilde{f}, \tilde{g}\right\}_\pi$ is well defined.

Therefore, giving a Poisson structure on a manifold is the same as giving a "generalized" folliation with symplectic leaves.

When $\pi$ is Poisson, $[\pi, \pi] = 0$, we can define

$$d_\pi = [\pi, \cdot] : C^\infty(\wedge^k T) \to C^\infty(\wedge^{k+1} T)$$

Note that $[\pi, \cdot]$ is of degree $(2-1)$, so it makes sense to cal it $d_\pi$. Also,

$$d^2_\pi(A) = [\pi, [\pi, A]] = [[\pi, \pi], A] - [\pi, [\pi, A]] = -[\pi, [\pi, A]]$$

$$\Longrightarrow d^2_\pi = 0$$

Thus, we have a chain complex

$$\ldots \longrightarrow C^\infty(\wedge^{k-1}T) \xrightarrow{d_\pi} C^\infty(\wedge^k T) \xrightarrow{d_\pi} C^\infty(\wedge^{k+1} T) \longrightarrow \ldots$$

Moreover, if $m_f$ denotes multiplication by $f \in C^\infty(M)$,

$$[d_\pi, m_f]\,\psi = d_\pi(f\psi) - f d_\pi \psi = [\pi, f\psi] - f[\pi, \psi] = [\pi, f] \wedge \psi = \iota_{df}\pi \wedge \psi$$

But for any $\xi \in T^*$, $\xi \neq 0$, $(\iota_\xi \pi)\wedge : \wedge^k T \to \wedge^{k+1} T$ is exact only for $\iota_\xi\pi \neq 0$. So, if $\pi$ is not invertible, $d_\pi$ is not an elliptic complex, and the Poisson cohomology groups, $H^k_\pi(M) = \text{Ker } d_\pi|_{\wedge^k T}/\text{Im } d_\pi|_{\wedge^{k-1}T}$ could be infinite dimensional on a compact $M$.

Let us look at the first such groups:

$H^0_\pi(M) = \{f : d_\pi f = 0\} = \{f : X_f = 0\} = \{$Casimir functions, i.e. functions s.t. $\{f, g\} = 0$ for all $g\}$

$H^1_\pi(M) = \{X : d_\pi X = 0\}/\text{Im } d_\pi = \{$infinitesimal symmetries of Poisson vector fields$\}$/Hamiltonians

$H^2_\pi(M) = \left\{P \in C^\infty(\wedge^2 T) : [\pi, P] = 0\right\} =$ tangent space to the moduli space of Poisson structures