15 Lecture 20 (Notes: K. Venkatram)

15.1 Generalized Complex Branes (of rank 1)

In complex geometry, we have special submanifolds, i.e. complex submanifolds $\phi : S \to M$ s.t. $J(TS) \subset TS$, i.e. $TS \subset TM$ is a complex subspace (for examplex, points in a manifold, or algebraic subvarieties). In symplectic geometry, there are several kinds of special submanifolds: isotropic ($TS \subset TS^{\omega}$), coisotropic ($TS^{\omega} \subset TS$), and Lagrangian ($TS = TS^{\omega} \Leftrightarrow \phi^*\omega = 0$).

Example.

1. If $f : (M, \omega) \to (M, \omega)$ is a diffeomorphism with $f^*\omega = \omega$ (i.e. a symplectomorphism), then $\phi : \Gamma_f \to M \times \overline{M}$ satisfies $\phi^*(\pi_1^*\omega - \pi_2^*\omega) = 0$, i.e. Γ_f is Lagrangian.
2. For any manfold M, T^*M is symplectic, with $\omega = \sum dp_i \wedge dx_i$, for $\{x_i\}$ a coordinate chart on M and $\{p_i\}$ coordinates for the 1-form. Then the fibers ($x_i = 0$) are Lagrangian, as are the zero sections ($p_i = 0$). Aimilarly, the graph of any 1-form $\alpha = \sum \alpha_i dx^i \in \Omega^1(M)$ is Lagrangian $\Leftrightarrow f^*\omega = \sum d\alpha_i \wedge dx^i = 0 \Leftrightarrow d\alpha = 0$.

Lagrangians and complex submanifolds are important in physics since they are the D-branes in A- and B-models. However, for a generalized complex manifold, we don't yet have such a good notion of subobject. Now, associated to any submanifold $S \to M$, we can form

$$0 \to TS \to TM \to NS \to 0 \tag{115}$$

and hence

$$0 \to N^*S \to T^*M \to T^*S \to 0 \tag{116}$$

where $N^*S = \{\xi \in T^*M | \xi(TS) = 0\}$ is the conormal bundle. Therefore, we have a natural maximal isotropic subbundle $TS \oplus N^*S \subset TM \oplus T^*M$. If there is ambient flux, i.e. (M, H), then as we defined before, $(f : S \to M, F \in \Omega^2(S))$ gives us a *topological brane* when $f^*H = dF$. In this case, we similarly have $\tau_{S,F} = f_*\Gamma_F \subset TM \oplus T^*M$ s.t.

$$f_*\Omega_F = \{f_*v + \xi \in TS \oplus T^*M | v + f^*\xi \in \Gamma_F\} \tag{117}$$

This gives us an exact sequence

$$0 \to N^*S \to \tau_{S,F} \to TS \to 0 \tag{118}$$

, and we call it a generalized complex brane when $\mathbb{J}\tau_{S,F} \subset \tau_{S,F}$.

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15.1.1 General Properties of Generalized Complex Branes

- $(f: S \to (M,H), F \in \Omega^2(S))$ has generalized pullback map $e^F f^* : \Omega^*(M) \ni \rho \mapsto e^F f^*\rho \in \Omega^*(S)$ s.t.

$$de^F f^*\rho = dF \wedge e^F f^*\rho + e^F f^* d\rho = e^F f^*(d\rho + H \wedge \rho) = e^F f^* d_H \rho \tag{119}$$

  Thus, we obtain a map on cohomology $H^*_H(M,\mathbb{R}) \to H^*_H(S,\mathbb{R})$.

- Let ψ be the pure spinor line in $\bigwedge^* T^*M|_S$ defining $\tau_{M,S}$ Then $\psi = \langle e^{-F}\det\,(N^*)\rangle$ and $\mathbb{J}\tau \subset \tau$ implies that

$$0 = (\mathbb{J}x)\psi = [\mathbb{J}, x]\cdot\psi = \mathbb{J}(x\cdot\psi) + x\cdot\mathbb{J}\cdot\psi \forall x \in \tau \tag{120}$$

  Thus, $\mathbb{J}\psi = (ik)\psi$: since ψ is real, $k=0$, and $\psi \in \mathcal{U}^0$.

- Gerbe interpretation: for $G = (L_{ij}, m_{ij}, \theta_{ijk})$ a gerbe, (∇_{ij}, B_i) a connection, if we can find (L_i, ∇_i) on S s.t. $F(\nabla_i) - F(\nabla_j) = F(\nabla_{ij})$, then $F(\nabla_i) - B_i = F$ is the gloabl 2-form on S we described.

- Action by B-fields: $e^B \circlearrowright T \oplus T^*, (S, F+B)$.

Example. Examples of generalized complex branes:

1. Complex Case: $f : (S,F) \to (M,J)(H=0)$. Then

$$\begin{aligned}&\tau_{S,F} = \{v + \xi \in TS \oplus T^*M | i_V F = f^*\xi\}\\ &\mathbb{J}\tau_{S,F} = \tau_{S,F} \Leftrightarrow J(TS) \subset TS \text{ and } -J^*Fv = FJv \Leftrightarrow S \text{ is a complex submanifold and } F \text{ has type } (1,1)\end{aligned} \tag{121}$$

   Thus, we interpret $F = F(\nabla)$ as the curvature of a unitary connection on a holomorphic line bundle $\mathcal{L}$, giving us the complex brane $(S,\mathcal{L},\nabla)$.

2. Symplectic Case: For $H = 0, F = 0$, we have

$$\mathbb{J}' = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}\begin{pmatrix} TS \\ N^*S\end{pmatrix} = \begin{pmatrix} TS \\ N^*S\end{pmatrix} \Leftrightarrow \omega(TS) = N^*S \text{ and } \omega^{-1}(N^*S) = TS \Leftrightarrow TS \subset TS^\omega \text{ and } TS^\omega \subset TS \tag{122}$$

   i.e. iff S is Lagrangian. For $F \neq 0$, things are more interesting. Choose locally an extension of F to $\Omega^2(M)$. Then $\mathbb{J}_\omega$ fixes $\tau_{S,F} \Leftrightarrow e^F \mathbb{J}_\omega e^{-F}$ fixed $\tau_{S,0} \Leftrightarrow$

$$\begin{pmatrix} -\omega^{-1}F & -\omega^{-1} \\ \omega + F\omega^{-1}F & F\omega^{-1}\end{pmatrix}\begin{pmatrix} TS \\ N^*S\end{pmatrix} = \begin{pmatrix} TS \\ N^*S\end{pmatrix} \tag{123}$$

   That is, we must have

   - $\omega^{-1}N^*S \subset TS$, i.e. S is coisotropic
   - $F(TS^\omega) \subset N^*S$, i.e. F vanishes on the characteristic foliation $\mathcal{C}$, i.e. locally $F = \pi^*\{, \pi : S \to S/\mathcal{C}$.
   - $\omega^{-1}F \circlearrowright TS$ s.t. $(\omega + F\omega^{-1}F)TS \subset N^*S)$, i.e. on $TS/TS^\omega, (1 + \omega^{-1}F\omega^{-1}F) = 0$, i.e. $(\omega^{-1}F)^2 = -1$. Thus, TS/TS^ω inherits a complex structure.

   Note that $F + i\omega$ defines a form of type $(2,0)$ on TS/TS^ω w.r.t. $I = \omega^{-1}F$ since

$$I^*(F + i\omega) = F\omega^{-1}(F + i\omega) = -\omega + iF = i(F + i\omega) = (F + i\omega)I \tag{124}$$

   and $F + i\omega$ is closed. Thus, $F + i\omega$ defines a holomorphic symplectic structure on $S\mathcal{C}$, which therefore must be $4k$-dimensional. This is precisely the geometry discovered by Kapustin and Orlov as the most general rank 1 A-brane in a symplectic manifold.

Example. Let (g, I, J) be a hyper-K ahler manifold, and consider the complex structure ω_I.

Example. If $S = M$, then the conditions are $(\omega^{-1}F)^2 = -1$, i.e. $F + i\omega$ is a holomorphic symplectic structure. For example, (M, g, I, J) hyperk ahler with $\omega = \omega_k, F = \omega_J, \omega^{-1}F = \omega_J^{-1}\omega_k = (gJ)^{-1}gk = -I$. This is an example of a space-filling rank 1 A-brane used by Kapustin-Witten in their study of the geometric Langlands program.

15.1.2 Branes for Other Generalized Complex Manifolds

Consider a complex structure I, deformed by a holomorphic bivector β: $Q = \beta + \overline{\beta}$, $\mathbb{J} = \begin{pmatrix} I & Q \\ & -I^* \end{pmatrix}$ is a generalized complex strucutre, e.g. $\mathbb{C}P^2$.

0-Branes: Before deformation, all the points were branes. Now, only the points on $\beta = 0$ are.

2-Branes: Branes must be complex curves where $\beta = 0$ or $(\beta + i\omega)$-Langrangian where $\beta \neq 0$. That is, $\beta = 0$ is a brane, as is any curve on which $\beta + i\omega = \beta^{-1}$ vanishes. In particular, any previous complex curve is still a brane.

Problem. Are there 2-branes in $\mathbb{C}P^2_\beta$ which are not complex curves in $\mathbb{C}P^2$? What are the space-filling branes on $\mathbb{C}P^2_\beta$?