16 Lecture 21-23 (Notes: K. Venkatram)

16.1 Linear Algebra

We define a category $\mathcal{H}$ whose objects are pairs (E, g) (sometimes denoted E for brevity), where E is a finite dimensional vector space $/\mathbb{R}$ and g is a nondegenerate symmetric bilinear form on E with signature 0, and whose morphisms are maximal isotropies $L \subset \overline{E} \times F$. Here, $E \mapsto \overline{E} = (E, -g)$ is the natural involution, and $E \times F = (E \times F, g_E + g_F)$ is the natural product structure. Composition is done by composition of relations, i.e. $E \to^L F \to^M G, M \circ L = \{(e, g) \in E \times G | \exists f \in F s.t. (e, f) \in L, (f, g) \in M\}$.

Proposition 11. *$M \circ L$ is a morphism in $\mathcal{H}$.*

Proof. $\mathcal{L} : L \times M \subset \overline{E} \times F \times \overline{F} \times G = W$ is maximally isotropic. $\mathcal{C} = E \times \Delta_F \times G$, where $\Delta_F = \{(f, f) | f \in F\}$, is coisotropic, i.e. $\mathcal{C}^\perp = \Delta_F \subset \mathcal{C}$. Thus, we get an induced bilinear form on $\mathcal{C}^\perp / \mathcal{C} = \overline{E} \times G$. $\mathcal{C} \cap \mathcal{L} + \mathcal{C}^\perp$ is maximaly isotropic in W, so

$$(\mathcal{C} \cap \mathcal{L} + \mathcal{C}^\perp)^\perp = (\mathcal{C}^\perp + \mathcal{L}^\perp) \cap \mathcal{C} = \mathcal{C}^\perp + \mathcal{L} \cap \mathcal{C} \tag{125}$$

Thus, $\mathcal{C} \cap \mathcal{L} + \mathcal{C}^\perp / \mathcal{C}^\perp = M \circ L \subset \mathcal{C}/\mathcal{C}^\perp = \overline{E} \times G$ is maximally isotropic. □

Remark. This cateogory is the symmetric version of the Weinstein's symplectic category ζ where $\mathrm{Ob}(\zeta) = (E, \omega)$ and morphisms are given by Lagrangians. Thus, is the the "odd" version or parity reversal of ζ.

A particular case of a morphism $E \to F$ is the graph of an orthogonal morphism.

Problem. Show that $L : E \to F$ is epi $\Leftrightarrow \pi_F(L) = F$, mono $\Leftrightarrow \pi_E(L) = E$, and iso $\Leftrightarrow L$ is orthogonal iso $E \to F$.

So for dim $E = 2n, O(n, n) \subset \mathrm{Hom}(E, E)$ are isos. But $\mathrm{Hom}(E, E) \cong O(2n)$ as a space since we can choose a positive definite C_+ and then any $L \in O(2n)$. This implies that $\mathrm{Hom}(E, E)$ is a monoid compactifying the group $O(E)$.

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16.1.1 Doubling Functor

Now, there is a nature "Double" functor $\mathcal{D} : \text{Vect} \to \mathcal{H}$ which maps $V \mapsto V \oplus V^*$ and $\{f : V \to M\} \mapsto \{\mathcal{D}f = \{(v + F^*\eta, f_*v + \eta) \in V \oplus V^* \times W \oplus W^* | v \in V, \eta \in W^*\}\}$. Note that $\mathcal{D}f \subset \overline{\mathcal{D}V} \times \mathcal{D}W$ and $\dim \mathcal{D}f = \dim V + \dim W$.

$$\langle (v + f^*\eta, f_*v + \eta), (v + f^*\eta, f_*v + \eta)\rangle = -f^*\eta(v) + \eta(f_*v) = 0 \tag{126}$$

Problem. Prove that $\mathcal{D}$ is a functor, i.e. $\mathcal{D}(f \circ g) = \mathcal{D}f \circ \mathcal{D}g$.

Note that $\mathcal{H}$ has a duality functor $L \in \text{Hom}(E, F) \implies L^* \in \text{Hom}(F, E)$, where $L^* = \{(f, e) | (e, f) \in L\}$.

Problem. Show that $\mathcal{D}(f^*) = (\mathcal{D}f)^*$.

Problem. Prove that $\mathcal{D}$ preserves epis and monos.

16.1.2 Maps Induced by Morphisms

A morphism $L \in \text{Hom}(E, F)$ induces maps $L \circ - : \text{Hom}(X, E) \rightleftarrows \text{Hom}(X, F) : L^* \circ -$. A special case is $X = \{0\}$, in which $\text{Hom}(0, E) = \text{Dir}(E)$, so $L \in \text{Hom}(E, F)$ induces maps $L_* : \text{Dir}(E) \rightleftarrows \text{Dir}(F) : L^*$. If L is mono or epi, so is L_*. This recovers the pushforward and pullback of Dirac structures: for $f : V \to W$ a linear map, $\mathcal{D}f : \mathcal{D}V \to \mathcal{D}W$ a morphism we obtain maps $\mathcal{D}f_* : \text{Dir}(V) \rightleftarrows \text{Dir}(W) : \mathcal{D}f^*$. As observed earlier, any Dirac $L \subset V \oplus V^*$ with $\pi_V(L) = M \subset V$ can be written as $L(M, B), B \in \bigwedge^2 M^*$, i.e. $L = j_*\Gamma_B$ for $j : M \hookrightarrow V$ the embedding and a unique B. That is, $L = j_*e^B M$.

Example. Given $f : V \to W$ a linear map, $\mathcal{D}f \subset \overline{\mathcal{D}V} \times \mathcal{D}W = \mathcal{D}(V \oplus W^*)$. and $\mathcal{D}f = ((v, f^*\eta), (f_*v, \eta) \cdots)$, hence $\pi_{V \oplus W^*}\mathcal{D}f = V \oplus W^*$ is onto. Therefore, $\mathcal{D}f = e^B(V \oplus W^*)$, and in fact $B = f \in V^* \otimes W \subset \bigwedge^2(V \oplus W^*)^*$.

16.1.3 Factorization of Morphisms $L : \mathcal{D}V \to \mathcal{D}(W)$

Let $L \in \text{Hom}(\mathcal{D}V, \mathcal{D}W), L \subset \overline{\mathcal{D}V} \times \mathcal{D}W \cong \mathcal{D}(V \oplus W)$. Then $L = j_*e^F M$, for $M = \pi_{V \oplus W} L \subset V \oplus W$. Let $\phi : M \to V, \psi : M \to W$ be the natural projections.

Theorem 13. *$L = \mathcal{D}\psi_* \circ e^F \circ \mathcal{D}\phi^*$.*

Proof. (Exercise) □

Corollary 10. *L is an isomorphism $\Leftrightarrow \phi, \psi$ are surjective and F determines a nondegenerate pairing* $\text{Ker } \phi \times \text{Ker } \psi \to \mathbb{R}$.

Therefore, an orthogonal map $V \oplus V^* \to W \oplus W^*$ can be viewed as a subspace $M \subset V \times W, F \in \bigwedge^2 M^*$.

16.2 T-duality

The basic idea of T-duality is as follows: let $S^1 \to P \to^\pi B$ be a principal S^1 bundle, i.e. a spacetime with geometry, with an invariant 3-form flux $H \in \Omega^3_{cl}(P)^{S^1}$ and an integral $[H] \in H^3(P, \mathbb{Z})$, i.e. coming from a gerbe with connection. Then we are going to produce a new "dual" spacetime with "isomorphic quantized field theory" (in this case, a sigma model). Specifically, let $\tilde{P}$ be a new S^1-bundle over B so that $c_1(\tilde{P}) = \pi_*(H) \in H^2(B, \mathbb{Z})$, and choose $\tilde{H} \in H^3(\tilde{P}, Z)$ s.t. $\tilde{\pi}_*\tilde{H} = c_1(P)$. More specifically, choose a connection $\theta \in \Omega^1(P)$ (i.e. $L_{\partial_\theta}\theta = 0, i_{\partial_\theta} = 1/2\pi$) so $d\theta = F \in \Omega^2(B)$ is integral and $[F] = c_1(P)$. Then $H = \tilde{F} \wedge \theta + h$ for some $\tilde{F} \in \Omega^2(B)$ integral and $H \in \Omega^3(B)$. Now, $[\tilde{F}] \in H^2(B, \mathbb{Z})$ defines a new principal S^1-bundle $\tilde{P}$. Choose a connection $\tilde{\theta}$ on $\tilde{P}$ so that $d\tilde{\theta} = \tilde{F}$. Then define $\tilde{H} = F \wedge \tilde{\theta} + h$, so tat $\int \tilde{H} = F$ and $\int H = \tilde{F}$.

Example. Let $S^1 \times S^2 \to S^2$ be the trivial S^1-bundle, with $H = v_1 \wedge v_2$. Then $v_2 = \int_{S^1} H = c_1(S^3 \to S^2)$, so the T-dual is the pair $S^3, 0$. Our original space has trivial topology and nontrivial flux, while the new space has nontrivial topology and trivial flux.

Remark. In physics, T-dual spaces have the same quantum physics, hence the same D-branes and twisted K-theory.

Theorem 14 (BHM)**.** *We have an isomorphism* $K^*_H(P) \cong K^{*+1}_{\tilde{H}}(\tilde{P})$.

Next, let $P \times_B \tilde{P} = \{(p, \tilde{p}) | \pi(p) = \tilde{\pi}(\tilde{p})\} \subset P \times \tilde{P}$ be the correspondence space, ϕ, ψ the two projections. Then $\phi^* H - \psi^* \tilde{H} = \tilde{F} \wedge \theta - F \wedge \tilde{\theta} = -d(\phi^* \theta \wedge \psi^* \tilde{\theta})$.

Definition 23. *A* T-duality *between* S^1*-bundles* (P, H) *and* $(\tilde{P}, \tilde{H})$ *over* B *is a 2-form* $F \in \Omega^2(P \times_B \tilde{P})^{S^1 \times S^1}$ *s.t.* $\phi^* H - \psi^* \tilde{H} = dF$ *and* F *deterines a nondegenerate pairing* $\mathrm{Ker}\ \phi_* \times \mathrm{Ker}\ \psi_* \to \mathbb{R}$.

In fact, T-duality can be expressed, therefore, as an orthogonal isomorphism

$$(T_p \oplus T_p^*, H)/S^1 \to^{L(P \times_B \tilde{P}, F)} (T_{\tilde{P}} \oplus T^*_{\tilde{P}}, \tilde{H})/S^1 \tag{127}$$

though of as bundles over B (or just S^1-invariant sections on $P, \tilde{P}$). This map sends H-twisted bracket to $\tilde{H}$-twisted bracket, via

$$\Omega^*(P)^{S^1} \ni \rho \mapsto \tau(\rho) = \psi_* e^F \wedge \phi^* \rho = \int_{\tilde{S}^1} e^F \wedge \phi^* \rho \in \Omega^*(\tilde{P})^{S^1} \tag{128}$$

Since $d(e^F \rho) = e^F(d\rho + (H - \tilde{H})\rho)$, we find that $d_{\tilde{H}}(e^F \rho) = e^F d_H \rho$ and $\tau(d_H \rho) = d_{\tilde{H}} \tau(\rho)$ as desired. Overall, a T-duality $F : (P, H) \to (\tilde{P}, \tilde{H})$ implies an isomorphism $(T_p \oplus T_p^*, H)/S^1 \to^{L(P \times_B \tilde{P}, F)} (T_{\tilde{P}} \oplus T^*_{\tilde{P}}, \tilde{H})/S^1$ as Courant algebroid, and thus any S^1-invariant generalized structure may be transported from (P, H) to $(\tilde{P}, \tilde{H})$.

Example.

1. $T^*_P \subset (T_p \oplus T_p^*, H)$ is a Dirac structure $\implies$ T-dual is

$$\tau(\xi + \theta) = \xi - \tilde{\partial}_\theta = T^* B + \langle \partial_{\tilde{\theta}} \rangle = \Delta \oplus \mathrm{Ann}\ \Delta \tag{129}$$

   for $\delta = \langle \partial_{\tilde{\theta}} \rangle$

2. The induced map on twisted cohomology $H^*_H(P) \rightleftarrows H^{*+1}_{\tilde{H}}(\tilde{P})$ is an isomorphism.

3. Where does τ take the subspace $C_+ = \Gamma_{g+b} \subset T^* \oplus T$? In $TP = TB \oplus 1$, decompose $g = g_0 \theta \odot \theta + g_1 \odot \theta + g_2, b = b_1 \wedge \theta + b_2$ for g_i, b_i basic. Then

$$C_+ = \Gamma_{g+b} = \langle x + f\partial_\theta + (i_x g_2 + f g_1 + i_x b_2 - f b_1) + (g_1(x) + f g_0 + b_1(x))\theta \rangle \tag{130}$$

   which is mapped via τ to

$$\Gamma_{\tilde{g}+\tilde{b}} = \langle x + (g_1(x) + f g_0 + b_1(x))\partial_{\tilde{\theta}} + (i_x g_1 + f g_1 + i_x b_2 - f b_1) + f\tilde{\theta} \rangle \tag{131}$$

   where

$$\begin{cases} \tilde{g} = \frac{1}{g_0} \tilde{\theta} \odot \tilde{\theta} - \frac{b_1}{g_0} \odot \tilde{\theta} + g_2 + \frac{1}{g_0}(b_1 \odot b_1 - g_1 \odot g_1) \\ \tilde{b} = \frac{-g_1}{g_0} \wedge \tilde{\theta} + b_2 + \frac{g_1 \wedge b_1}{g_0} \end{cases} \tag{132}$$

   These are called "Buscher rules".

4. Elliptic Curves: