3 Lecture 3 (Notes: J. Bernstein)

3.1 Almost Complex Structure

Let $J \in \mathbb{C}^\infty(\text{End}(T))$ be such that $J^2 = -1$. Such a J is called an *almost complex structure* and makes the real tangent bundle into a complex vector bundle via declaring $iv = J(v)$. In particular dim $_{\mathbb{R}}M = 2n$. This also tells us that the structure group of the tangent bundle reduces from $Gl(2n, \mathbb{R})$ to $Gl(n, \mathbb{C})$. Thus T is an associated bundle to a principal $Gl(n, \mathbb{C})$ bundle. In particular we have map on the cohomology,

$$\begin{aligned} H^{2i}(M, \mathbb{Z}) &\rightarrow H^{2i}(M, \mathbb{Z}/2\mathbb{Z}) \\ c(T, J) &\mapsto w(T) \end{aligned}$$

Where $c(T, J)$ are the *Chern classes* of T (with complex structure given by J) and $w(T)$ are the *Stiefel-Whitney classes*. Here the map is reduction mod 2. In particular $w_{2i+1} = 0$ and $c_1 \mapsto w_2$, the later fact implies that M is *Spin*^{c}.

Recall that the *Pontryagin classes* of a vector bundle are $p_i \in H^{4i}$ such that $p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C})$. We study $p_i(T) = (-1)^i c_{2i}(T \otimes \mathbb{C})$. Since the eigenvalues of $J : T \rightarrow T$ are $\pm i$ we have the natural decomposition

$$T \otimes \mathbb{C} = (\text{Ker } (J - i)) \oplus (\text{Ker } (J + i)) = T_{1,0} \oplus T_{0,1}$$

Here $T_{1,0}$ and $T_{0,1}$ are complex subbundles of $T \otimes \mathbb{C}$ and on has the identifications $(T_{1,0}, i) \cong (T, J)$ and $(T_{0,1}, i) \cong (T, -J)$. Hence if we choose a hermitian metric h on T we get a non degenerate pairing,

$$T_{1,0} \times T_{0,1} \rightarrow \mathbb{C}$$

and hence $T_{1,0} \cong (T_{0,1})^*$. We now compute

$$\sum_k (-1)^k p_k(T) = \sum_k c_{2k}(T_{1,0} \oplus T_{0,1}) = \sum_k \sum_i c_i(T_{1,0}) \cup c_{2k-i}(T_{0,1}) = (\sum_i c_i(T_{1,0})) \cup (\sum_j c_j(T_{0,1})$$

where the last equality comes from rearranging the sum. Now we have $c_i(T_{0,1}) = (-1)^i c_i(T_{1,0})$ and since we can identity $T_{1,0}$ with (T, J) we have

$$\sum_k (-1)^k p_k(T) = (\sum_i c_i(T, J)) \cup (\sum_j (-1)^j c_j(T, J))$$

Thus the existence of an almost complex structure implies that one can find classes $c_i \in H^{2i}(M, \mathbb{Z})$ that when taken mod 2 give the Stiefel-Whitney class and that satisfy the above Pontryagin relation.

Problem. Show that S^{4k} does not admit an almost complex structure.

Remark. Topological obstructions to the existence of an almost complex structure in general are not known.

3.2 Hermitian Structure

Definition 10. *A* hermitian structure *or a real vector space V consists of a triple*

- *J an almost complex structure*
- *$\omega : V \rightarrow V^*$ ω symplectic (i.e. $\omega^* = -\omega$)*
- *$g : V \rightarrow V^*$ g a metric (i.e. $g^* = g$ and if we write $x \mapsto g(x, \cdot)$ then $g(x, x) > 0$ for $x \neq 0$)*

with the compatibility

$$g \circ J = \omega$$

Now pick (J, g) this determines a hermitian structure if and only if

$$-(gJ) = (gJ)^* = J^* g^* = J^* g$$

. On the other hand (J, ω) determines a hermitian structure if and only if

$$-(\omega J) = (\omega J^{-1})^* = -J^* \omega^* = J^* \omega$$

that is if and only if $J^*\omega + \omega J = 0$. Then we have $(J^*\omega + \omega J)(v)(w) = \omega(Jx, y) + \omega(x, Jy) = 0$ which is equivalent to ω of type (1,1). We get three structure groups

$$\begin{array}{rcl} g & \mapsto & O(V, g) = \{A : A^* g A = g\} \\ \omega & \mapsto & Sp(V, \omega) = \{A^* \omega A = \omega\} \\ J & \mapsto & Gl(V, J) = \{A : AJ = JA\} \end{array}$$

Now if we form $h = g + i\omega$ we obtain a hermitian metric on V. And we have structure group

$$\mathrm{Stab}(h) = U(V, h) = O(v, h) \cap Sp(V, \omega) = Gl(V, J) \cap O(V, g) = Sp(V, \omega) \cap Gl(V, J)$$

we note $U(V, h)$ is the maximal compact subgroup of $Gl(V, J)$.

Problem. 1. Show Explicitly that given J one can always find a compatible ω (or g)
2. Show similarly that givne ω can find compatible g.

3.3 Integrability of J

Since we have a Lie bracket on T we can tensor it with $\mathbb{C}$ and obtain a Lie bracket on $T \otimes \mathbb{C}$. The since $T \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1}$, integrability conditions are thus that the complex distribution $T_{1,0}$ is involutive i.e. $[T_{1,0}, T_{1,0}] \subseteq T_{1,0}$. How far is this geometry from usual complex structure on $\mathbb{C}^n$? Idea is if one can form $M^{\mathbb{C}}$ the complexification of M (think of $\mathbb{R}P^n \subset \mathbb{C}P^n$ or $\mathbb{R}^n \subset \mathbb{C}^n$, indeed if M is real analytic it is always possible to do this. Then $M^{\mathbb{C}}$ has two transverse foliations by the integrability condition (from $T_{1,0}$ and $T_{0,1}$). Say functions $z^i : M^{\mathbb{C}} \to \mathbb{C}$ cut out the leaves of $T_{1,0}$ (i.e. the leaves are given by $z^1 = z^2 = \ldots = z^n = c$). Then when one restricts the $z^i i$ to a neighborhood $U \subseteq M$, obtains maps $z^1, \ldots, z^n : U \to \mathbb{C}$ such that $< dz^1, \ldots, dz^n >= T^*_{1,0} = Ann(T_{0,1}$. That is one obtains a holomorphic coordinate chart. Moreover in this chart one has

$$J = \sum_k i(dz^k \otimes \frac{\partial}{\partial z^k} + d\overline{z}^k \otimes \frac{\partial}{\partial \overline{z}^k})$$

Remark. This is similar to the Darboux theorem of symplectic geometry

More generally we have

Theorem 4. *(Newlander-Nirenberg) If M is a smooth manifold with smooth almost complex structure J that is integrable then M is actually complex.*

Note. This was most recently treated by Malgrange.

Now $T_{1,0}$ closed under $[,]$ happens if and only if for $X \in T, X - iJX \in T_{1,0}$ one has $[X - iJX, Y - iJY] = Z - iJZ$. That is $[X, Y] - [JX, JY] + J[X, JY] + J[JX, Y] = 0$

Definition 11. *We define the* Nijenhuis tensor *as* $N_J(X, Y) = [X, Y] - [JX, JY] + J[X, JY] + J[JX, Y]$

Problem. Show that N_J is a tensor in $C^\infty(\bigwedge^2 T^* \otimes T)$.

Thus one has J integrable if and only if $N_J = 0$.

Remark. N_J=0 is the analog of $d\omega \in C^\infty(\bigwedge^3 T^*)$

Now if we view $J \in \text{End}(T) = \Omega^1(T) = \sum \xi^i \otimes \nu_i$ then J acts on differential forms, $\rho \in \Omega^\cdot(M)$ by $\imath_J(\rho) = \sum \xi^i \wedge \imath_{v_i}\rho = \sum(e_{\xi^i} \cdot \imath_{v_i})\rho$. And one computes

$$\imath_J(\alpha \wedge \beta) = \imath_J(\alpha) \wedge \beta + (-1)^\alpha \alpha \wedge \imath_J \beta$$

thus $\imath_J \in \text{Der}^0(\Omega^\cdot(M))$ and we may form $L_J = [\imath_J, d] \in \text{Der}^1(\Omega^\cdot(M))$.

Note. L_J is denoted d^c

Definition 12. *We define the* Nijenhuis bracket $[,] : \Omega^k \times \Omega^l \to \Omega^{k+l}$ *by* $L_{[J,K]} = [L_J, L_K]$

One checks $[L_J, L_J] = L_{N_J}$ hence $N_J = [J, J]$.

3.4 Forms on a Complex Manifold

In a manner similar with our treatment of foliations, we wish to express integrability in terms of differentiable forms. Let $T_{0,1}$ (or $T_{1,0}$) be closed under the complexified Lie bracket. Since Ann $T_{0,1} = T^*_{1,0} =< \theta^1, \ldots, \theta^n >$ (Ann $T_{1,0} = T^*_{1,0}$), $\Omega = \theta^1 \wedge \ldots \theta^n$ is a generator for det $T^*_{1,0} = K$. Where here K is a complex line bundle. The condition for integrability is then $d\Omega^{n,0} = \xi^{0,1} \wedge \Omega^{n,0}$ for some ξ. Taking d again one obtains $0 = d\xi \wedge \Omega^{n,0} - \xi \wedge d\Omega = d\xi \wedge \Omega$, hence $\overline{\partial}\xi = 0$. We call $K = \bigwedge^n T^*_{1,0}$ the *canonical bundle.*

Note. This definition is deserved since $K \subset \bigwedge T^* \otimes \mathbb{C}$ and $T_{0,1} = AnnK = \{X \, \imath_X \Omega = 0\}$, i.e. we can recover the complex structure from K

More fully, there is a decomposition of forms

$$\bigwedge^{\cdot} T^* \otimes \mathbb{C} = \bigoplus_{p,q} \left(\bigwedge^p T^*_{1,0} \bigotimes \bigwedge^q T^*_{0,1} \right)$$

$$\Omega^\cdot = \bigoplus_{p,q} \Omega^{p,q}(M)$$

that is a $\mathbb{Z} \times \mathbb{Z}$ grading.

Since $d\Omega^{n,0} = \xi \wedge \Omega$ we have integrability if and only if $d = \partial + \overline{\partial}$, where here $\partial = \pi_{p,q+1} \circ d$ and $\overline{\partial} = \pi_{p+1,q} \circ d$.

Problem. Show that without integrability

$$d = \partial + \overline{\partial} + d^N$$

where $N_J \in \wedge^2 T^* \otimes T$ and $d^N = \imath_{N_J}$. Also determine the p, q decomposition of d^N.

3.5 Dolbeault Cohomology

Assuming $N_J = 0$ one has $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$. Thus one gets a complex

$$\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

The cohomology of this complex is called the *Dolbeault cohomology* and is denoted

$$\frac{\text{Ker } \overline{\partial}|_{\Omega^{p,q}}}{\text{Im } \overline{\partial}|_{\Omega^{p,q-1}}} = H^{p,q}_{\overline{\partial}}(M).$$

This is a $\mathbb{Z}\times\mathbb{Z}$ graded ring. The symbol of $\overline{\partial}$ can be determined from the computation $[\overline{\partial}, m_f] = e_{\overline{\partial} f}$. Now given a real form $\xi \in T^* - \{0\}$ then

$$\begin{array}{ccc} \bigwedge^{p,q} T^* & \to & \bigwedge^{p,q+1} T^* \\ \rho & \mapsto & \xi^{0,1} \wedge \rho \end{array}$$

is elliptic, since $\xi = \xi^{1,0} + \xi^{0,1} = \xi^{1,0} + \overline{\xi^{0.1}}$ (as ξ real) and so $\xi^{0,1} \neq 0$. Hence dim $H^{p,q}_{\overline{\partial}} < \infty$ on M compact.

Now suppose $E \to M$ is a complex vector bundle, how does pone make E compatible with the complex structure J on M?

Definition 13. *$E \to M$ a complex vector bundle is a* holomorphic *if there exists a connection $\overline{\partial}_E : C^\infty(E) \to C^\infty(T^*_{0,1} \otimes E)$ which is flat (i.e. $\overline{\partial}^2_E = 0$).*

This gives us a complex

$$C^\infty(T^*_{0,1} \otimes E) \to \ldots \to \Omega^{0,q}(E) = C^\infty(\wedge^{0,q} T^* \otimes E) \to \ldots$$

The cohomology of this complex is called *Dolbeault cohomology with values in E* and is denoted $H^q_{\overline{\partial}_E}(M, E)$. Elliptic theory tells us that M compact implies $H^q_{\overline{\partial}_E}(M, E)$ is finite dimensional. We note that $\overline{\partial}|_{\Omega^{n,0}}$ is a holomorphic structure on K and hence K is a holomorphic line bundle.

Problem. Find explicitly the $\overline{\partial}_E$ operator on $E = T_{1,0}$