5 Lecture 5 (Notes: C. Kottke)

5.1 Spinors

We have a natural action of $V \oplus V^*$ on $\bigwedge^{\cdot} V^*$. Indeed, if $X + \xi \in V \oplus V^*$ and $\rho \in \bigwedge^{\cdot} V^*$, let

$$(X + \xi) \cdot \rho = i_X \rho + \xi \wedge \rho.$$

Then

$$\begin{aligned} (X+\xi)^2 \cdot \rho &= i_X(i_X\rho + \xi \wedge \rho) + \xi \wedge (i_X \rho + \xi \wedge \rho) \\ &= (i_X \xi)\rho - \xi \wedge i_X \rho + \xi \wedge i_X \rho \\ &= \langle X + \xi, X + \xi \rangle \rho \end{aligned}$$

where $\langle , \rangle$ is the natural symmetric bilinear form on $V \oplus V^*$:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)).$$

Thus we have an action of $v \in V \oplus V^*$ with $v^2 \rho = \langle v, v \rangle \rho$. This is the defining relation for the Clifford Algebra $CL(V \oplus V^*)$.

For a general vector space E, $CL(E, \langle , \rangle)$ is defined by

$$CL(E, \langle , \rangle) = \bigotimes E / \langle v \otimes v - \langle v, v \rangle 1 \rangle$$

That is, $CL(E, \langle , \rangle)$ is the quotient of the graded tensor product of E by the free abelian group generated by all elements of the form $v \otimes v - \langle v, v \rangle 1$ for $v \in E$. Note in particular that if $\langle , \rangle \equiv 0$ then $CL(E, \langle , \rangle) = \bigwedge^{\cdot} E$.

We therefore have representation $CL(V \oplus V^*) \xrightarrow{\cong} \mathrm{End}(\bigwedge^{\cdot} V^*) \cong \mathrm{End}(\mathbb{R}^{2^n})$ where $n = \dim V$. This is called the "spin" representation for $CL(V \oplus V^*)$.

Choose an orthonormal basis for $V \oplus V^*$, i.e. $\{e_1 \pm e^1, \ldots, e_n \pm e^n\}$. The clifford algebra has a natural volume element in terms of this basis given by

$$\omega \equiv (-1)^{\frac{n(n-1)}{2}} (e_1 - e^1) \cdots (e_n - e^n)(e_1 + e^1) \cdots (e_n + e^n).$$

Problem. Show $\omega^1 = 1$, $\omega e_i = -e_i \omega, \omega e^i = -e^i \omega$, and $\omega \cdot 1 = 1$, considering 1 as the element in $\bigwedge^0 V^*$ acted on by the clifford algebra.

The eigenspace of ω is naturally split, and we have

$$S^+ \equiv \mathrm{Ker}(\omega - 1) = \bigwedge^{\mathrm{ev}} V^*$$
$$S^- \equiv \mathrm{Ker}(\omega + 1) = \bigwedge^{\mathrm{od}} V^*$$

The e^i are known as "creation operators" and the e_i as "annihilation operators". We define the "spinors" S by

$$S = \bigwedge^{\cdot} V^* = S^+ \oplus S^-$$

Here is another view. V is naturally embedded in $V \oplus V^*$, so we have

$$CL(V) = \bigwedge^{\cdot} V \subset CL(V \oplus V^*)$$

since $\langle V, V \rangle = 0$. Note in particular that $\det V \subset CL(V \oplus V^*)$, where $\det V$ is generated by $e_1 \cdots e_n$ in terms of our basis elements. $\det V$ is a minimal ideal in $CL(V \oplus V^*)$, so $CL(V \oplus V^*) \cdot \det V \subset CL(V \oplus V^*)$. Elements of $CL(V \oplus V^*) \cdot \det V$ are generated by elements which look like

$$\underbrace{(1, e^i, e^i e^j, \ldots)}_{\text{no } e_i} \quad \underbrace{e_1 \cdots e_n}_{\equiv f \in \det V}$$

For $x \in CL(V \oplus V^*)$ and $\rho \in S$, the action $x \cdot \rho$ satisfies $x \rho f = (x \cdot \rho) f$.

Problem. Show that this action coincides with the Cartan action.

5.2 The Spin Group

The spin group $\text{Spin}(V \oplus V^*) \subset CL(V \oplus V^*)$ is defined by

$$\text{Spin}(V \oplus V^*) = \{v_1 \cdots v_r : v_i \in V \oplus V^*, \langle v_i, v_i \rangle = \pm, r \text{ even.}\}$$

$\text{Spin}(V \oplus V^*)$ is a double cover of the special orthogonal group $\text{SO}(V \oplus V^*)$; there is a map

$$\rho : \text{Spin}(V \oplus V^*) \xrightarrow{2:1} \text{SO}(V \oplus V^*)$$

where the action $\rho(x) \cdot v = xvx^{-1}$ in $CL(V \oplus V^*)$.

The adjoint action in the Lie algebra $\mathfrak{s}o(V \oplus V^*)$ is given by

$$d\rho_x : v \longmapsto [x, v]$$

where $[,]$ is the commutator in $CL(V \oplus V^*)$, so

$$\mathfrak{s}o(V \oplus V^*) = \text{span}\{[x, y] : x, y \in V \oplus V^*\} \cong \bigwedge^2 (V \oplus V^*).$$

Recall that $\bigwedge^2 (V \oplus V^*) = \bigwedge^2 V^* \oplus \bigwedge^2 V \oplus \text{End}(V)$, so a generic element in $\bigwedge^2 (V \oplus V^*)$ looks like

$$B + \beta + A \in \bigwedge^2 V^* \oplus \bigwedge^2 V \oplus \text{End}(V)$$

In terms of the basis, say $B = B_{ij} e^i \wedge e^j$, $\beta^{ij} e_i \wedge e_j$, and $A = A_i^j e^i \otimes e_j$. In $CL(V \oplus V^*)$, these become $B_{ij} e^i e^j$, $\beta^{ij} e_j e_i$ and $\frac{1}{2} A_i^j (e_j e^i - e^i e_j)$, respectively. Consider the action of each type of element on the spinors.

$$(B_{ij} e^i e^j) \cdot \rho = B_{ij} e^i \wedge e_i \wedge \rho = -B \wedge \rho$$

$$(\beta^{ij} e_j e_i) \cdot \rho = \beta^{ij} i_{e_j} i_{e_i} \rho = i_\beta \rho$$

$$\left(\frac{1}{2} A_i^j (e_j e^i - e^i e_j) \right) \cdot \rho = \frac{1}{2} A_i^j (i_{e_j} (e^i \wedge \rho) - e^i \wedge i_{e_j} \rho) = (\frac{1}{2} A_i^j \delta_j^i) \rho - A_i^j e^i \wedge e_j \rho = \left(\frac{1}{2} \text{Tr} A \right) \rho - A^* \rho$$

Given $B \in \bigwedge^2 V^*$, recall the B field transform e^{-B}. This acts on the spinors via

$$e^{-B} \cdot \rho = \rho + B \wedge \rho + \frac{1}{2!} B \wedge B \wedge \rho + \cdots$$

Note that there are only finitely many terms in the above.

Similarly, given $\beta \in \bigwedge^2 V$, we have

$$e^\beta \cdot \rho = \rho + i_\beta \rho + \frac{1}{2} i_\beta i_\beta \rho + \cdots$$

For $A \in \text{End}(V)$, $e^A \equiv g \in \text{GL}^+(V)$, we have

$$g \cdot \rho = \sqrt{\det(g)} \left(g^{*-1} \right) \cdot \rho$$

so that, as a $\text{GL}^+(V)$ representation, $S \cong \bigwedge^{\cdot} V^* \otimes (\det V)^{1/2}$.

5.3 A Bilinear Pairing on Spinors

Let $\rho, \phi \in \bigwedge^{\cdot} V^*$ and consider the reversal map $\alpha : \bigwedge^{\cdot} V^* \to \bigwedge^{\cdot} V^*$ where

$$\xi_1 \wedge \cdots \wedge \xi_k \overset{\alpha}{\longmapsto} \xi_k \wedge \cdots \wedge \xi_1$$

Define

$$(\rho, \phi) = [\alpha(\rho) \wedge \phi]_n \in \det V^*$$

where $n = \dim V$, and the subscript n on the bracket indicates that we take only the degree n parts of the resulting form.

Proposition 3. *For $x \in CL(V \oplus V^*)$, $(x \cdot \rho, \phi) = (\phi, \alpha(x) \cdot \phi)$*

Proof. Recall that $(x \cdot \rho)f = x\rho f$ and

$$\begin{aligned}(\rho, \phi) &= i_f(\rho, \phi)f \\ &= i_f(\alpha(\rho) \wedge \phi))f \\ &= \alpha(f)\alpha(\rho)\phi f \\ &= \alpha(\rho f)\phi f\end{aligned}$$

so $(x \cdot \rho, \phi) = \alpha(x\rho f)\phi f = \alpha(\rho f)\alpha(x)\phi f = (\rho, \alpha(x)\phi)$. □

Corollary 2. *We have*

$$(v \cdot \rho, v \cdot \phi) = (\rho, \alpha(v)v \cdot \phi) = \langle v, v\rangle(\rho, \phi)$$

Also, for $g \in Spin(V \oplus V^*)$,

$$(g \cdot \rho, g \cdot \phi) = \pm 1(\rho, \phi)$$

Example. Suppose $n = 4$, and $\rho, \phi \in \bigwedge^{\mathrm{ev}} V^*$, so that

$$\rho = \rho_0 + \rho_2 + \rho_4$$

and similarly for ϕ, where the subscripts indicate forms of degree 0, 2, and 4. Then $\alpha(\rho) = \rho_0 - \rho_2 + \rho_4$ and

$$(\rho, \phi) = \left[(\rho_0 - \rho_2 + \rho_4) \wedge (\phi_0 + \phi_2 + \phi_4)\right]_4 = \rho_0\phi_4 + \phi_0\rho_4 - \rho_2 \wedge \phi_2$$

If $n = 4$ and $\rho, \phi \in \bigwedge^{\mathrm{od}} V^*$, then

$$(\rho, \phi) = \left[(\rho_1 - \rho_3) \wedge (\phi_1 + \phi_3)\right]_4 = \rho_1 \wedge \phi_3 - \rho_3 \wedge \phi_1.$$

Proposition 4. *In general, $(\rho, \phi) = (-1)^{\frac{n(n-1)}{2}}(\phi, \rho)$*

Problem.

- What is the signature of $(,)$ when symmetric?
- Show that $(,)$ is non-degenerate on $S^{\pm}$.
- Show that in dimension 4, the 16 dimensional space $\bigwedge^{\cdot} V^*$ has a non degenerate symmetric form

5.4 Pure Spinors

Let $\phi \in \bigwedge^{\cdot} V^*$ be any nonzero spinor, and define the null space of ϕ as

$$L_\phi = \{X + \xi \in V \oplus V^* : (X + \xi) \cdot \phi = 0\}.$$

It is clear then that L_ϕ depends equivariantly on ϕ under the spin representation. If

$$\phi \mapsto g \cdot \phi, \qquad g \in \mathrm{Spin}(V \oplus V^*)$$

then

$$L_\phi \mapsto \rho(g) L_\phi$$

where $\rho : \mathrm{Spin}(V \oplus V^*) \to \mathfrak{so}(V \oplus V^*)$ as before. The key property of the null space is that it is isotropic. Indeed, if $x, y \in L_\phi$ we have

$$2\langle x, y\rangle \phi = (xy + yx)\phi = 0.$$

Thus $L_\phi \subset L_\phi^\perp$.

If $L_\phi = L_\phi^\perp$, that is, if L_ϕ is maximal, then ϕ is called "*pure*". We have therefore that ϕ is pure if and only if L_ϕ is Dirac.

Example.

- Take $\phi = e^1 \wedge \cdots \wedge e^n$. Then $L_\phi = V^*$.
- Take $1 \in \bigwedge^0 V^*$. Then $L_1 = V$. For $B \in \bigwedge^2 V^*$, then $e^{-B} \cdot 1 = 1 - B + 1/2 B \wedge B + \cdots$. So $L_{e^B} = e^B(L_1) = e^B(V) = \Gamma_B$.
- For $\theta \in V^*$, θ is pure since $L_\theta = \{X + \xi : i_X\theta + \xi \wedge \theta = 0\} = \mathrm{Ker}\ \theta \oplus \langle\theta\rangle$ which is Dirac; indeed this is what we called $L(\mathrm{Ker}\ \theta, 0)$.
- Similarly, considering $e^B\theta$, we have $L_{e^B\theta} = L(\mathrm{Ker}\ \theta, f^*B)$.
- Given a Dirac structure $L(E, \epsilon)$, choose $\theta_1, \ldots, \theta_k$ such that $\langle\theta_1, \ldots, \theta_k\rangle = \mathrm{Ann}\ E$. Choose $B \in \bigwedge^2 V^*$ such that $f_\epsilon^* B = \epsilon$. Then $\phi = e^{-B}\theta_1 \wedge \cdots \wedge \theta_k$ is pure and $L_\phi = L(E, \epsilon)$.

Problem.

- Show $L_\phi \cap L_{\phi'} = \{\emptyset\} \Leftrightarrow (\phi, \phi') \neq 0$.
- Let $\dim V = 4$, and $\rho = \rho_0 + \rho_2 + \rho_4 \neq 0$. Show that ρ is pure iff $2\rho_0\rho_4 = \rho_2 \wedge \rho_2$.