6 Lecture 6 (Notes: Y. Lekili)

Recall from last lecture :

$S = \Lambda^\bullet V^*, (X+\xi)\cdot\rho = \iota_X\rho + \xi\wedge\rho$. Mukai pairing $(\rho,\phi) = [\rho\wedge\alpha(\phi)]_n$ Spin_0-invariant.

$$\mathrm{Dir}(V) \longleftrightarrow \text{Pure spinors}$$
$$L_\phi \longleftrightarrow \phi = ce^B\theta_1\wedge\ldots\wedge\theta_k, k = \text{type}$$

Problem.

1. Prove that $L_\phi \cap L'_\phi = \{0\} \Leftrightarrow (\phi,\phi') \neq 0$
2. Let $\dim V = 4$. Show that $0 \neq \rho = \rho_0 + \rho_2 + \rho_4$ is pure iff $2\rho_0\rho_4 = \rho_2\wedge\rho_2$. Show in general dimension that Pur = Pure spinors $\subset S^\pm$ are defined by a quadratic cone. Indentify the space $\mathbb{P}(Pur) \subset \mathbb{P}(S^\pm.)$

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6.1 Generalized Hodge star

C_+ positive definite. $C_+ : V \to V^*$, $C_+(X)(X) > 0$ for $X \neq 0$. $C_+ = \Gamma_{g+b}$, $g \in S^2V^*$ and $b \in \Lambda^2 V^*$. Note that C_+ determines an operator

$$G : V \oplus V^* \to V \oplus V^*$$

$\langle Gx, Gy\rangle = \langle x, y\rangle, G^2 = 1$. So $G^* = G$. G is called a *generalized metric* since $\langle Gx, y\rangle$ is positive definite. Note that if $C_+ = \Gamma_g : \{v + g(v)\}$ and $C_- = \{v - g(v)\}$ then $G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$. In general $C_+ = \Gamma_{g+b} = e^b\Gamma_g$ so

$$G = e^b \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-b} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}$$

Problem. Note that restriction of G to T is $g - bg^{-1}b$. Verify that it is indeed positive definite.

Comment about the volume form of $g - bg^{-1}b = g^b$:

Note: $g - bg^{-1}b = (g - b)g^{-1}(g + b)$. So $\det(g - bg^{-1}b) = \det(g - b)\det(g^{-1})\det(g + b)$, and $\det(g + b) = \det(g + b)^* = \det(g - b)$. Hence $vol_{g^b} = \det(g - bg^{-1}b)^{1/2} = \frac{\det(g+b)}{\det(g)^{1/2}}$.

Problem. What is vol_{g^b}/vol_g ?

Aside: $\det V^*$, choose orientation. $\det V^* \otimes V^*$, natural orientation since square. $\det g(v \otimes v) > 0$ so $\det g$ has square roots. After choice of orientation on V, there exists a unique positive square root vol_g.

A generalized metric is given by $G : V \oplus V^* \to V \oplus V^*$ such that $G^2 = 1, G^* = G, \langle G(x), x\rangle > 0$.
$C_\pm = ker(G \mp 1)$.
Consider $* = a_1 \dots a_n$ where $(a_1, \dots, a_n)$ is an oriented basis for C_+. $* \in \mathrm{CL}(C_+) \subset \mathrm{CL}(V \oplus V^*)$.

- $*$ is the volume element of $\mathrm{CL}(C_+)$
- $*$ is the lift of $-G$ to $\mathrm{Pin}(V \oplus V^*) = \{v_1 \dots v_r : ||v_i|| = \pm 1\}$ (Spin if n is even)
- $*$ acts on forms via $* \cdot \rho = a_1 \dots a_n \cdot \rho$.

Consider $b = 0$ and e_i, e^i orthonormal basis. Then $* = (e_1 + e^1)\dots(e_n + e^n)$. Consider $\alpha(*) = (e_n + e^n)\dots(e_1 + e^1)$. $\alpha(*)1 = e^n \wedge \dots \wedge e^1$, $\alpha(*)e^1 = e^n \wedge \dots \wedge e^2$, $\dots etc$. So,

$$\alpha(\alpha(*)\rho) = \star\rho, \text{ Hodge star.}$$

So $\alpha(\alpha(*)\rho)$ is generalized Hodge star. Note that $*^2 = (-1)^{\frac{n(n-1)}{2}}$ and $(\rho, \phi) = (-1)^{\frac{n(n-1)}{2}}(\phi, \rho)$. So consider $(*\rho, \phi)$ is symmetric pairing of ρ, ϕ into $\det V^*$. And note that if $b = 0$,

$$(*\rho, \phi) = (\rho, \alpha(*)\phi) = [\rho \wedge \alpha(\alpha(*)\phi)]_{top} = [\rho \wedge \star\phi]_{top} = g(\rho, \phi)vol_g$$

When $b \neq 0, G = e^b \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-b}$. So $* = e^b *_g e^{-b}$, and $(*\rho, \phi) = (e^b *_g e^{-b}\rho, \phi) = (*_g(e^{-b}\rho)e^{-b}\phi)$. So always nondegenerate for all b. Hence $(*\rho, \phi) = G(\rho, \phi)(*1, 1)$ with $G(1, 1) = 1$ where G is the natural symmetric pairing on forms.

Problem. Let $e_1, \dots, e_n$ be g-orthonormal basis of V.

- Show $(e_i + (g + b)(e_i))$ form orthonormal basis of C_+.
- Show $(*1, 1) = det(g + b)(e_1 \wedge \dots \wedge e_n) = \frac{\det(g+b)}{\det(g)^{1/2}} = vol_{g^b}$
- As a result, show $\frac{vol_{g^b}}{vol_g} = ||e^{-b}||^2_g$

6.2 Spinors for $TM \oplus T^*M$ and the Courant algebroid

On a manifold M, $T = TM$, $T^* = T^*M$. $T \oplus T^*$ is a bundle with $\langle , \rangle$ structure $O(n,n)$. $S = \Lambda^\bullet T^*$.

$$\text{Diff forms} \longleftrightarrow \text{Spinors for } T \oplus T^*.$$

New element: $d : \Omega^k \to \Omega^{k+1}$. Recall $[X,Y]$ is defined by $\iota_{[X,Y]} = [L_X, \iota_Y] = [[d, \iota_X], \iota_Y]$. We now use same strategy to define a bracket on $T \oplus T^*$.

$$(X + \xi) \cdot \rho = (\iota_X + \xi \wedge)\rho$$

So for $e_1, e_2 \in C^\infty(T \oplus T^*)$, define

$$[[d, e_1 \cdot], e_2 \cdot]\rho = [e_1, e_2]_{\mathcal{C}} \cdot \rho$$

the Courant bracket on $C^\infty(T \oplus T^*)$. Note $[d, \iota_X + (\xi\wedge)] = L_X + (d\xi \wedge)$ and

$$[L_X + (d\xi\wedge), \iota_Y + (\eta\wedge)] = \iota_{[X,Y]} + ((L_X\eta)\wedge) - ((\iota_Y d)\xi\wedge).$$

Hence

$$[[d, e_1 \cdot], e_2 \cdot]\rho = \iota_{[X,Y]}\rho + (L_X\eta - \iota_Y d\xi) \wedge \rho$$

defines a bracket, Courant bracket :

$$[X + \xi, Y + \eta] = [X,Y] + L_X\eta - \iota_Y d\xi.$$

Note bracket is not skew-symmetric: $[X + \xi, X + \xi] = L_X\xi - \iota_X d\xi = d\iota_X\xi = d\langle X + \xi, X + \xi\rangle$. It is skew "up to exact terms" or "up to homotopy". However, it does satisfy Jacobi identity:

$$[[a,b].c] = [a,[b,c]] - [b,[a,c]].$$

Proof: $[d, \cdot] = D$ an inner graded derivation on $\mathrm{End}\Omega$. $D^2 = 0$. $[a,b]_{\mathcal{C}} \cdot \phi = [[d,a],b] \cdot \phi = [Da, b]$ Then $[[a,b]_{\mathcal{C}}, c]_{\mathcal{C}} \cdot \phi = [D[Da,b],c]\phi = [[Da,Db],c]\phi = [Da,[Db,c]] - [Db,[Da,c]] = [a,[b,c]_{\mathcal{C}}]_{\mathcal{C}} - [b,[a,c]_{\mathcal{C}}]_{\mathcal{C}}$.

It is also obviously compatible with Lie bracket.

$$T \oplus T^* \xrightarrow{\pi} T$$
$$[\ ,\ ]_{\mathcal{C}} \longrightarrow [\ ,\ ]$$

that is, $[\pi a, \pi b] = \pi[a,b]_{\mathcal{C}}$.

Two main key properties :

- $[a, fb] = f[a,b] + ((\pi a)(f))b$.

  Let $a = X + \xi, b = Y + \eta$,
  $[X+\xi, f(Y+\eta)] = [X, fY] + L_X(f\eta) - f\iota_Y d\xi = f[a,b] + (Xf)Y + (Xf)\eta = f[X+\xi, Y+\eta] + (Xf)(Y+\eta)$.

- How does it interact with $\langle , \rangle$? $\pi a\langle b,b\rangle = 2\langle [a,b], b\rangle$

  $\langle [a,b], b\rangle = \iota_{[X,Y]}\eta + \iota_Y(L_X\eta - \iota_Y d\xi) = L_X\iota_Y\eta = \frac{1}{2}L_X\langle b,b\rangle = \pi a\langle b,b\rangle$

Usually written : $\pi a\langle b,c\rangle = \langle [a,b],c\rangle + \langle b,[a,c]\rangle$.
This defines the notion of *Courant Algebroid*:
$(E, \langle,\rangle, [,], \pi)$ where E is a real vector bundle, $\pi : E \to T$ is called anchor, $\langle,\rangle$ is nondegenerate symmetric bilinear form, $[,] : C^\infty(E) \times C^\infty(E) \to C^\infty(E)$ such that :

- $[[e_1,e_2],e_3] = [e_1,[e_2,e_3]] - [e_2,[e_1,e_3]]$
- $[\pi e_1, \pi e_2] = \pi[e_1,e_2]$
- $[e_1, fe_2] = f[e_1,e_2] + (\pi e_1)f)e_2$
- $\pi e_1\langle e_2,e_3\rangle = \langle [e_1,e_2],e_3\rangle + \langle e_2,[e_1,e_3]\rangle$
- $[e_1,e_1] = \pi^* d\langle e_1,e_1\rangle$

E is exact when

$$0 \to T^* \xrightarrow{\pi^*} E \xrightarrow{\pi} T \to 0$$

So $T \oplus T^*$ is exact Courant algebroid.
This motivates *Lie Algebroid*: $A \xrightarrow{\pi} T$, $[,] : C^\infty(A) \times C^\infty(A) \to C^\infty(A)$ Lie alg. such that

- $\pi[a,b] = [\pi a, \pi b]$
- $[a, fb] = f[a,b] + ((\pi a)f)b$