7 Lecture 7 (Notes: N. Rosenblyum)

7.1 Exact Courant Algebroids

Recall that a Courant algebroid is given by the diagram of bundles

$$\begin{array}{ccc} E & \xrightarrow{\pi} & T \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

where π is called the “anchor” along with a bracket $[\ ,\ ]$ and a nondegenerate bilinear form $\langle\ ,\ \rangle$ such that

- $\pi[a,b] = [\pi a, \pi b]$
- The Jacobi identity is zero
- $[a, fb] = f[a,b] + ((\pi a)f)b$
- $[a,b] = \frac{1}{2}\pi^* d\langle a,a\rangle$
- $\pi a\langle b,c\rangle = \langle [a,b],c\rangle + \langle b,[a,c]\rangle$

A Courant algebroid is exact if the sequence

$$0 \longrightarrow T^* \xrightarrow{\pi} E \xrightarrow{\pi^*} T \longrightarrow 0$$

is exact (note that $\pi \circ \pi^*$ is always 0).
Remarks: For an exact Courant algebroid, we have:

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1. The inclusion $T^* \subset E$ is automatically isotropic because for $\xi, \eta \in T^*$,

$$\langle \pi^*\xi, \pi^*\eta \rangle = \xi(\pi^*\pi\eta) = 0$$

since $\langle \pi^*\xi, a\rangle = \xi(\pi a)$.

2. The bracket $[\ ,\ ]|_{T^*} = 0$: for $s, t \in C^\infty(E)$, $f \in C^\infty(M)$,

$$\mathcal{D} = \pi^* d : C^\infty(M) \to C^\infty(E)$$

Now,

$$\langle [s, \mathcal{D}f], t\rangle = \pi s\langle \mathcal{D}f, t\rangle - \langle \mathcal{D}f, [s,t]\rangle = \pi s(\pi t(f)) - \pi[s,t](f) = \pi t(\pi s(f)) = \langle \mathcal{D}\langle \mathcal{D}f, s\rangle, f\rangle$$

Thus, $[s, \mathcal{D}f] = \mathcal{D}\langle s, \mathcal{D}f\rangle$. We also have, $[\mathcal{D}f, s] + [s, \mathcal{D}f] = \mathcal{D}\langle \mathcal{D}f, s\rangle$ and therefore $[Df, s] = 0$. We need to show that $[fdx^i, gdx^j] = 0$. But have $[dx^i, dx^j] = 0$ and

$$[a, fb] = f[a,b] + ((\pi a)f)b, \quad [ga, b] = g[a,b] - ((\pi b)g)a + 2\langle a, b\rangle dg.$$

7.2 Ševera's Classification of Exact Courant Algebroids

We can choose an isotropic splitting

$$0 \longrightarrow T^* \underset{s^*}{\overset{\pi^*}{\rightleftarrows}} E \underset{s}{\overset{\pi}{\rightleftarrows}} T \longrightarrow 0$$

i.e. $\langle sX, sY\rangle = 0$ for all $X, Y \in T$. We then have $E \cong T \oplus T^*$ and we can transport the Courant structure to $T \oplus T^*$: for $X, Y \in T$ and $\xi, \eta \in T^*$,

$$\langle X + \xi, Y + \eta\rangle = \langle sX + \pi^*\xi, sY + \pi^*\eta\rangle = \xi(\pi s Y) + \eta(\pi s X) = \xi(Y) + \eta(X)$$

since $\langle sX, sY\rangle = 0$. Also,

$$[X + \xi, Y + \eta] = [sX + \pi^*\xi, sY + \pi^*\eta] = [sX, sY] + [sX, \pi^*\eta] + [\pi^*\xi, sY]$$

.
We have that the second term is given by

$$\pi[sX, \pi^*\eta] = [\pi s X, \pi\pi^*\eta] = 0$$

and therefore, $[sX, \pi^*\eta] \in \Omega^1$. Further,

$$[sX, \pi^*\eta](Z) = \langle [sX, \pi^*\eta], sZ\rangle = X\langle \pi^*\eta, sZ\rangle - \langle \pi^*\eta, [sX, sZ]\rangle = X\eta(Z) - \eta([X,Z]) = i_Z L_X \eta$$

and so $[sX, \pi^*\eta] = L_X\eta$.
Now, the third term is given by

$$\langle [\pi^*\xi, sY], sZ\rangle = \langle -[sY, \pi^*\xi] + \mathcal{D}\langle sY, \pi^*\xi\rangle, sZ\rangle = -(L_Y\xi)(Z) + i_Z d i_Y \xi = (-i_Y d\xi)(Z)$$

and so $[\pi^*\xi, sY] = -i_Y d\xi$.
For the first term, we have no reason to believe that $[sX, sY] = [X, Y]$ We do have that $\pi[sX, sY] = [X, Y]_{Lie}$. Now, let $H(X, Y) = s^*[sX, sY]$. We then have,

1. H is C^∞-linear and skew in X, Y:

$$H(X, fY) = fs^*[sX, sY] + s^*(X(f)sY) = fs^*[sX, sY], \text{ and}$$

$$H(fX, Y) = s^*[fsX, sY] = fH(X, Y) - s^*((Yf)sX) + 2\langle sX, sY\rangle df = fH(X, Y). \text{ Furthermore,}$$
$$[sX, sY] + [sY, sX] = \pi^* d\langle sX, sY\rangle.$$

2. $H(X, Y)(Z)$ is totally symmetic in X, Y, Z:

$$H(X, Y)(Z) = \langle [sX, sY], sZ\rangle_E = X\langle sY, sZ\rangle - \langle sY, [sX, sZ]\rangle$$

So, we have $[sX, sY] = [X, Y] - i_Y i_X H$ for $H \in \Omega^3(M)$.

Problem. Show that $[[a, b], c] = [a, [b, c]] - [b, [a, c]] + i_{\pi c} i_{\pi b} i_{\pi a} dH$ and so $Jac = 0$ if and only if $dH = 0$.

Thus, we have that the only parameter specifying the Courant bracket is a closed three form $H \in \Omega^3(M)$. We will see that when $[H]/2\pi \in H^3(M, \mathbb{Z})$, E is associated to an S^1-gerbe.
Now, let's consider how H changes when we change the splitting. Suppose that we have two section $s_1, s_2 : T \to E$. We then have that $\pi(s_1 - s_2) = 0$. So consider $B = s_1 - s_2 : T \to T^*$. In the s_1 splitting, we have for $x \in T$, $s_2(x) = (x + (s_2 - s_1)x)$. Since the s_i are isotropic splittings, we have that $(s_2 - s_1)(x)(x) = 0$. Thus we have, $B \in C^\infty(\Lambda^2 T^*)$.
Now, in the s_1 splitting we have,

$$[X + i_x B, Y + i_Y B]_H = [X, Y] + L_X i_Y B - i_Y d i_X B + i_Y i_X H = [X, Y] + i_{[X,Y]} B - i_Y L_X B + i_Y d i_X B + i_Y i_X H =$$

$$= [X, Y] + i_{[X,Y]} B + i_Y i_X (H + dB)$$

In particular, in the s_2 splitting H changes by dB. Thus, we have that $[H] \in H^3(M, \mathbb{R})$ classifies the exact Courant algebroid up to isomorphis.
The above bracket is also a derived bracket. Before, we had that

$$[a, b]_{\mathcal{C}} \cdot \varphi = [[d, a], b]\varphi.$$

Now, replace d with $d_H = d + H\wedge$. We clearly have that $d_H^2 = (dH)\wedge = 0$ since $dH = 0$. Note that d_H is not of degree one and is not a derivation but it is odd. The cohomology of d_H is called H-twisted deRham cohomology. In simple cases (e.g. when M is formal in the sense of rational homotopy theory,), we have

$$H^*(H^{ev/od}(M), e_{[H]}) = H^{ev/od}_{d_H}(M)$$

where $e_H = H\wedge$.
Now, $[a, b]_H \cdot \varphi = [[d_H, a], b]\varphi$. Indeed, for $B \in \Omega^2$, we have $\varphi \mapsto e^B \varphi$ and $e^{-B}(d + H\wedge)e^B = e^{-B} de^B + e^{-B} He^B = d_{H+dB}$, and so $e^B[e^{-B}\cdot, e^B\cdot]_H = [\ ,\ ]_{H+dB}$ In particular, if $B \in \Omega^2_c l$, then e^B is a symmetry of the Courant bracket.
This phenomena is somewhat unusual because for the ordinary Lie bracket, the only symmetries are given by diffeomorphisms of the underlying manifold. More specifically, a symmetry of the Lie bracket on $C^\infty(T)$ is a diagram

$$\begin{array}{ccc} T & \xrightarrow{\Phi} & T \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & M \end{array}$$

such that ϕ is a diffeomorphism and $[\Phi, \Phi] = \Phi[\cdot, \cdot]$.

Claim 1. *$Sym[\ ,\ ]_{Lie} = \{(\phi_*, \phi),\ \phi \in Diff(M)\}$.*

Proof. Given $(\Phi, \phi) \in Sym[\ ,\ ]_{Lie}$, consider $G : \Phi\phi_*^{-1}$. Then G covers the identity map on M and we have $fG[X,Y] - ((Yf)GX = G[fX,Y] = f[GX,GY] - (GY)fGX$ and so $Yf = (GY)(f)$ for all Y, f and so $G = 1$. □

Let's now consider the question of what all the symmetries of the Courant bracket $[\ ,\ ]_{\mathcal{C}}$ are. Once again, we have a diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & M \end{array}$$

where $E \simeq T \oplus T^*$ such that

1. $\phi^* \langle \Phi \cdot, \Phi \cdot \rangle = \langle \cdot, \cdot \rangle$
2. $[\Phi \cdot, \Phi \cdot] = \Phi[\cdot, \cdot]$
3. $\pi \circ \Phi = \phi_* \circ \pi$.

Suppose that $\phi \in Diff(M)$. Then on $T \oplus T^*, \phi_*$ is given by

$$\phi_* = \begin{pmatrix} \phi_* & \\ & (\phi^*)^{-1} \end{pmatrix}$$

and so we have $\phi_*(X + \xi) = \phi_* X + (\phi^*)^{-1}\xi$ and

$$\phi_*^{-1}[\phi_* X + (\phi^*)^{-1}\xi, \phi_* Y + (\phi^*)^{-1}\eta]_H = [X + \xi, Y + \eta]_{\phi^* H}$$

since $\phi_*^{-1}(i_{\phi_* Y} i_{\phi_* X} H)(Z) = i_{\phi_* Z} i_{\phi_* Y} i_{\phi_* X} H = \phi^* H(X, Y, Z)$. In particular, this does not give a symmetry unless $\phi^* H = H$.
Now, consider a B-field transform. Since $e^B[e^{-B}\cdot, e^{-B}\cdot]_H = [\cdot, \cdot]_{H+dB}$, this is not a symmetry unless $dB = 0$. Now we can combine these to generate the symmetries:

$$[\phi_* e^B \cdot, \phi_* e^B \cdot] = \phi_* e^B [\cdot, \cdot]_{\phi^* H + dB}$$

and so $\phi_* e^B \in SymE$ iff $H - \phi^* H = dB$. It turns out that these are all the symmetries.

Theorem 5. *The above are all the symmetries of an exact Courant algebroid. In particular, we have a short exact sequence*

$$0 \to \Omega^2_{cl} \to Sym(E) \to Diff_{[H]} \to 0$$

where $Diff_{[H]}$ is the subgroup of diffeomorphisms of M preserving the cohomology class $[H]$.