18.969 Topics in Geometry: Mirror Symmetry
Spring 2009

MIRROR SYMMETRY: LECTURE 2

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Reference for today: M. Gross, D. Huybrechts, D. Joyce, "Calabi-Yau Manifolds and Related Geometries", Chapter 14.

1. Deformations of Complex Structures

An (almost) complex structure (X, J) splits the complexified tangent and (wedge powers of) cotangent bundles as

$$\begin{aligned} TX \otimes \mathbb{C} &= TX^{1,0} \oplus TX^{0,1}, v^{0,1} = \frac{1}{2}(v + iJv) \\ T^*X \otimes \mathbb{C} &= T^*X^{1,0} \oplus T^*X^{0,1}, T^*X^{1,0} = \mathrm{Span}(dz_i), T^*X^{0,1} = \mathrm{Span}(d\bar{z}_i) \\ \bigwedge^k T^*X \otimes \mathbb{C} &= \bigoplus_{p+q=k} \bigwedge^{p,q} T^*X = \Omega^{p,q}(X) \end{aligned} \tag{1}$$

If J is almost complex, these are $\mathbb{C}$-vector bundles. J is integrable (i.e. a complex structure)

$$\begin{aligned} [T^{1,0}, T^{1,0}] \subset T^{1,0} &\Leftrightarrow d = \partial + \bar{\partial} \text{ maps } \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1} \\ &\Leftrightarrow \bar{\partial}^2 = 0 \text{ on diff. forms} \end{aligned} \tag{2}$$

We obtain a Dolbeault cohomology for holomorphic vector bundles E:

$$\begin{aligned} C^q_{\bar{\partial}}(X, E) &= \{C^\infty(X, E) \xrightarrow{\bar{\partial}} \Omega^{0,1}(X, E) \xrightarrow{\bar{\partial}} \Omega^{0,2}(X, E) \to \cdots\} \\ H^q_{\bar{\partial}}(X, E) &= \ker\bar{\partial}/\mathrm{im}\bar{\partial} \end{aligned} \tag{3}$$

Deforming J to a "nearby" J' gives

$$\Omega^{1,0}_{J'} \subseteq T^*\mathbb{C} = \Omega^{1,0}_J \oplus \Omega^{0,1}_J \tag{4}$$

is a graph of a linear map $(-s) : \Omega^{1,0}_J \to \Omega^{0,1}_J$. J' is determined by $\Omega^{1,0}_{J'}$ (acted on by i) and $\Omega^{0,1}_{J'}$ (acted on by i'). s is a section of $(\Omega^{1,0}_J)^* \otimes \Omega^{0,1}_J = \mathbb{T}^{1,0}_j \otimes \Omega^{0,1}_J$ i.e. a $(0,1)_J$-form with values in $T^{1,0}_J X$. If $z_1, \ldots, z_n$ are local holomorphic coordinates for J, then $s = \sum s_{ij} \frac{\partial}{\partial z_i} \otimes d\bar{z}_j$. A basis of $(1,0)$-forms for J' is given by $dz_i - \underbrace{\sum_j s_{ij} d\bar{z}_j}_{s(dz_i)}$ and $(0,1)$-vectors for J' by $\frac{\partial}{\partial \bar{z}_k} + \underbrace{\sum_\ell s_{\ell k} \frac{\partial}{\partial z_\ell}}_{s(\partial/\partial \bar{z}_k)}$.

We can use this to test the integrability of J'. The Dolbeault complex $(\bigoplus_q \Omega_X^{0,q} \otimes TX^{1,0}, \bar{\partial})$ ($\bar{\partial}$ acts "on forms") carries a Lie bracket

$$[\alpha \otimes v, \alpha' \otimes v'] = (\alpha \wedge \alpha') \otimes [v, v'] \tag{5}$$

giving it the structure of a *differential graded Lie algebra.*

Proposition 1. *J' is integrable* $\Leftrightarrow \bar{\partial} s + \frac{1}{2}[s,s] = 0$.

Proof. We want to check that the bracket of two $0,1$ tangent vectors is still $0,1$, i.e. that

$$[\frac{\partial}{\partial \bar{z}_k} + \sum_\ell s_{\ell k} \frac{\partial}{\partial z_\ell}, \frac{\partial}{\partial \bar{z}_k} + \sum_\ell s_{\ell k} \frac{\partial}{\partial z_\ell}] \in TX^{0,1}_{J'} \tag{6}$$

Evaluating this bracket gives

$$\sum_\ell \left(\frac{\partial s_{\ell j}}{\partial \bar{z}_i} - \frac{\partial s_{\ell i}}{\partial \bar{z}_j} \right) \frac{\partial}{\partial z_\ell} + \sum_{k,\ell} (s_{ki} \frac{\partial s_{\ell j}}{\partial z_k} - s_{kj} \frac{\partial s_{\ell i}}{\partial z_k}) \frac{\partial}{\partial z_\ell} \tag{7}$$

We want this to be 0, i.e. for all i, j, ℓ,

$$0 = \underbrace{\frac{\partial s_{\ell j}}{\partial \bar{z}_i} - \frac{\partial s_{\ell i}}{\partial \bar{z}_j}}_{\text{coefficient of } \frac{\partial}{\partial z_\ell} \otimes (d\bar{z}_i \wedge d\bar{z}_j) \text{ in } (\bar{\partial} s)} + \sum_k \underbrace{(s_{ki} \frac{\partial s_{\ell j}}{\partial z_k} - s_{kj} \frac{\partial s_{\ell i}}{\partial z_k})}_{\text{in } \frac{1}{2}[s,s]} \tag{8}$$

We leave the rest as an exercise. □

We would now like to use this to understand the moduli space of complex structures. Define

$$\mathcal{M}_{CX}(X) = \{J \text{ integrable complex structures on } X\}/\mathrm{Diff}(X) \tag{9}$$

(or, assuming that $\mathrm{Aut}(X, J)$ is discrete, we want that near J, $\exists$ a universal family $\mathcal{X} \to \mathcal{U} \subset \mathcal{M}_{CX}$ (complex manifolds, holomorphic fibers $\cong X$) s.t. any family of integrable complex structures $\mathcal{X}' \to S$ induces a map $S \to \mathcal{U}$ s.t. $\mathcal{X}$ pulls back to $\mathcal{X}'$). We have an action of the diffeomorphisms of X: for $\phi \in \mathrm{Diff}(X)$ close to id,

$$\begin{aligned} d\phi &: TX \otimes \mathbb{C} \xrightarrow{\sim} \phi^* TX \otimes \mathbb{C} \\ \partial\phi &: TX^{1,0} \to \phi^* TX^{1,0} \\ \bar{\partial}\phi &: TX^{0,1} \to \phi^* TX^{1,0} \end{aligned} \tag{10}$$

so

$$\begin{aligned} \phi^* dz_i &= dz_i \circ d\phi = dz_i \circ \partial\phi + dz_i \circ \overline{\partial}\phi \\ &= (\underbrace{dz_i \circ \partial\phi}_{(1,0) \text{ for } J})(\mathrm{id} \ + (\partial\phi)^{-1} \cdot \overline{\partial}\phi) \end{aligned} \tag{11}$$

Deformation by $s \in \Omega^{0,1}(X, TX^{1,0})$ gives $\Omega^{1,0}_{J'} = \{\alpha - s(\alpha) | \alpha \in \Omega^{1,0}\}$ (the graph of $-s$): taking $s = -(\partial\phi)^{-1} \cdot \overline{\partial}\phi : TX^{0,1} \to \phi^* TX^{1,0} \to TX^{1,0}$ gives the desired element of $\Omega^{0,1}(TX^{1,0})$.

1.1. **First-order infinitesimal deformations.** Given a family $J(t), J(0) = J$ gives $s(t) \in \Omega^{0,1}(X, TX^{1,0}), s(0) = 00$. By the above, this should satisfy

$$\overline{\partial} s(t) + \frac{1}{2}[s(t), s(t)] = 0 \tag{12}$$

In particular, $s_1 = \frac{ds}{dt}|_{t=0}$ solves $\overline{\partial} s_1 = 0$. We obtain an infinitesimal action of $\mathrm{Diff}(X)$: for $(\phi_t), \phi_0 = \mathrm{id}\ , \frac{d\phi}{dt}|_{t=0} = v$ a vector field,

$$\frac{d}{dt}|_{t=0}(-(\partial\phi_t)^{-1} \circ \overline{\partial}\phi_t) = -\frac{d}{dt}|_{t=0}(\overline{\partial}\phi_t) = -\overline{\partial} v \tag{13}$$

This implies that first-order deformations are given as

$$\mathrm{Def}_1(X, J) = \frac{\text{Ker } (\overline{\partial} : \Omega^{0,1}(TX^{1,0}) \to \Omega^{2,0}(TX^{1,0}))}{\mathrm{Im}(\overline{\partial} : C^\infty(TX^{1,0}) \to \Omega^{0,1}(TX^{1,0}))} \tag{14}$$

We can write this more compactly using Dolbeault cohomology, namely $H^1_{\overline{\partial}}(X, TX^{1,0})$. Furthermore, given a family

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & S \end{array} \tag{15}$$

of deformations of (X, J) parameterized by S, we get a map $T_* S \to H^1(X, TX^{1,0})$ called the *Kodaira Spencer map*

Remark. A complex manifold (X, J) is a union of complex charts U_i with biholomorphisms $\phi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ s.t. $\phi_{ij} = \phi_{ji}^{-1}$ and $\phi_{ij}\phi_{jk} = \phi_{ik}$ on U_{ijk}. Deformations of (X, J) come from deforming the gluing maps ϕ_{ij} among the space of holomorphic maps. To first order, this is given by holomorphic vector fields v_{ij} on $U_i \cap U_j$ s.t. $v_{ij} = -v_{ji}$ and $v_{ij} + v_{jk} = v_{ik}$ on U_{ijk}. This is precisely the Čech 1-cocycle conditions in the sheaf of holomorphic tangent vector fields. Modding out by holomorphic functions $\psi_i : U_i \xrightarrow{\sim} U_i$ (which act by $\phi_{ij} \mapsto \psi_j \phi_{ij} \psi_i^{-1}$) is precisely modding by the Čech coboundaries. Thus, $\mathrm{Def}_1(X, J) = \check{H}^1(X, TX^{1,0})$.

1.2. **Obstructions to Deformation.** Given a first-order deformation s_1, one can ask if one can find an actual deformation $s(t) = s_1 t + O(t^2)$ (or even a formal deformation, i.e. non-convergent power series). Expand

$$s(t) = s_1 t + s_2 t^2 + \cdots \in \Omega^{0,1}(X, TX^{1,0}) \tag{16}$$

Then the condition $\overline{\partial} s(t) + \frac{1}{2}[s(t), s(t)] = 0$ implies that $\overline{\partial} s_1 = 0, \overline{\partial} s_2 + \frac{1}{2}[s_1, s_1] = 0, \overline{\partial} s_3 + [s_1, s_2] = 0, \cdots$. Now, we need $[s_1, s_1] \in \text{im } (\overline{\partial}) \subset \Omega^{0,2}(TX^{1,0})$. We know that $[s_1, s_1] \in \text{Ker } (\overline{\partial})$. Thus, the primary obstruction to deforming is the class of $[s_1, s_1]$ in $H^2(X, TX^{1,0})$. If it is zero, then there is an s_2 s.t. $\overline{\partial} s_2 + \frac{1}{2}[s_1, s_1] = 0$, and the next obstructure is the class of $[s_1, s_2] \in H^2(X, TX^{1,0})$. We are basically attempting to apply by brute force the implicit function theorem.

If it happens that $H^2(X, TX) = 0$, then the deformations are unobstructed and the moduli space of complex structures is locally a smooth orbifold (not a manifold, because we may have to quotient by automorphisms) with tangent space $H^1(X, TX^{1,0})$. For Calabi-Yau manifolds, this will not be true: however, we still have

Theorem 1 (Bogomolov-Tian-Todorov)**.** *For X a compact Calabi-Yau ($\Omega_X^{n,0} \cong \mathcal{O}_X$) with $H^0(X, TX) = 0$ (automorphisms are discrete), deformations of X are unobstructed and, assuming* $\text{Aut}(X, J) = \{1\}$, *$\mathcal{M}_{CX}$ is locally a smooth manifold with $T\mathcal{M}_{CX} = H^1(X, TX)$.*

Theorem 2 (Griffiths Transversality)**.** *For a family (X, J_t), $\alpha_t \in \Omega^{p,q}(X, J_t) \implies \frac{d}{dt}|_{t=0} \alpha_t \in \Omega^{p,q} + \Omega^{p+1,q-1} + \Omega^{p-1,q+1}$.*

Proof. J_t is given by $s(t) \in \Omega^{0,1}(TX^{1,0}), s(0) = 0$. In local coordinates, we have $T^* X^{1,0}_{J_t} = \text{Span}\{dz_i^{(t)} = dz_i - \sum s_{ij}(t) d\overline{z}_j\}$

$$\alpha_t = \sum_{I,J||I|=p,|J|=q} \alpha_{IJ}(t) dz_{i_1}^{(t)} \wedge \cdots \wedge dz_{i_p}^{(t)} \wedge d\overline{z}_{j_1}^{(t)} \wedge \cdots \wedge d\overline{z}_{j_q}^{(t)} \tag{17}$$

Taking $\frac{d}{dt}|_{t=0}$, the result follows from the product rule. We mostly get (p, q) terms and a few $(p+1, q-1), (p-1, q+1)$ forms (the latter from $\frac{d}{dt}|_{t=0}(dz_{i_k}^{(t)})$. □