18.969 Topics in Geometry: Mirror Symmetry
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MIRROR SYMMETRY: LECTURE 16

DENIS AUROUX

0.1. **Coherent Sheaves on a Complex Manifold (contd.)** Let X be a complex manifold, $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Recall that the category of sheaves has both an internal $\mathscr{H}om$ (which is a sheaf) and an external Hom (the group of global sections for the former). A functor $F : \mathcal{C} \to \mathcal{C}'$ is left exact if $0 \to A \to B \to C \to 0 \implies 0 \to F(A) \to F(B) \to F(C)$. If the category $\mathcal{C}$ has enough injectives (objects such that $\mathrm{Hom}_C(-, I)$ is exact), there are right-derived functors R^iF s.t.

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots \tag{1}$$

To compute $R^iF(A)$, resolve A by injective objects as $0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$, we get a complex $0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots$. Taking cohomology gives $R^iF(A) = \mathrm{Ker}\,(F(I^i) \to F(I^{i+1}))/\mathrm{im}\,(F(I^{i-1}) \to F(I^i))$. Note that $R^0F(A) = F(A)$.

We stated last time that sheaf cohomology arises as the right-derived functor of the global sections functor. Moreover, since $\mathrm{Hom}(\mathcal{E}, -)$ and $\mathrm{Hom}(-, \mathcal{E})$ are both left-exact (the first covariant, the second contravariant), we can define $\mathrm{Ext}^i = R^i\mathrm{Hom}$, and short exact sequences $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ give

$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathcal{E}, \mathcal{F}_1) \to \mathrm{Hom}(\mathcal{E}, \mathcal{F}_2) \to \mathrm{Hom}(\mathcal{E}, \mathcal{F}_3) \\ &\to \mathrm{Ext}(\mathcal{E}, \mathcal{F}_1) \to \mathrm{Ext}(\mathcal{E}, \mathcal{F}_2) \to \mathrm{Ext}(\mathcal{E}, \mathcal{F}_3) \to \cdots \end{aligned} \tag{2}$$

while sequences $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ give

$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathcal{E}_3, \mathcal{F}) \to \mathrm{Hom}(\mathcal{E}_2, \mathcal{F}) \to \mathrm{Hom}(\mathcal{E}_1, \mathcal{F}) \\ &\to \mathrm{Ext}(\mathcal{E}_3, \mathcal{F}) \to \mathrm{Ext}(\mathcal{E}_2, \mathcal{F}) \to \mathrm{Ext}(\mathcal{E}_1, \mathcal{F}) \to \cdots \end{aligned} \tag{3}$$

Moreover, if $\mathcal{E}$ is a locally free sheaf, $\mathscr{H}om(\mathcal{E}, -)$ is exact, and $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) = H^i(\mathscr{H}om(\mathcal{E}, \mathcal{F}))$. Otherwise, we can resolve $\mathcal{E}$ by locally free sheaves

$$0 \to E_n \to \cdots \to E_0 \to \mathcal{E} \to 0 \tag{4}$$

and, for all practical purposes, replace $\mathcal{E}$ by the complex $E_n \to \cdots \to E_0$. In our case, we obtain a sequence $\mathscr{H}om(E_0, \mathcal{F}) \to \cdots \to \mathscr{H}om(E_n, \mathcal{F})$ whose hypercohomology gives $\mathrm{Ext}^*(\mathcal{E}, \mathcal{F})$.

Example. Let $\mathcal{E}$ be a locally free sheaf, $\mathcal{O}_p$ the skyscraper sheaf at a point p. Then $\mathscr{H}om(\mathcal{E},\mathcal{O}_p) \cong \mathcal{E}^*|_p$ is the skyscraper sheaf with stalk $\mathcal{E}_p^*$ at p. Taking sheaf cohomology gives $\mathrm{Hom}(\mathcal{E},\mathcal{O}_p) \cong \mathcal{E}_p^*, \mathrm{Ext}^i(\mathcal{E},\mathcal{O}_p) = 0 \,\forall\, i \geq 1$. Furthermore, $\mathscr{H}om(\mathcal{O}_p,\mathcal{O}_p) \cong \mathcal{O}_p$: to obtain the higher Ext groups, we resolve $\mathcal{O}_p$ by locally free sheaves. (WLOG) Assuming X is affine, local coordinates near p define a section s of $\mathcal{O}_X^{\oplus n} \cong V$ ($n = \dim X$) vanishing transversely at p. We then have a long exact sequence

$$0 \to \left(\bigwedge^n V^* \xrightarrow{s} \bigwedge^{n-1} V^* \xrightarrow{s} \cdots \xrightarrow{s} V^* \xrightarrow{s} \mathcal{O}_X\right) \to \mathcal{O}_p \to 0 \tag{5}$$

Applying $\mathscr{H}om(-,\mathcal{O}_p)$, we get

$$\mathcal{O}_p \xrightarrow{0} V \otimes \mathcal{O}_p \xrightarrow{0} \cdots \xrightarrow{0} \bigwedge^{n-1} V \otimes \mathcal{O}_p \xrightarrow{0} \bigwedge^n V \otimes \mathcal{O}_p \tag{6}$$

(the maps are all zero, since all the sheaves are all skyscraper sheaves at p). $\mathrm{Ext}^*(\mathcal{O}_p,\mathcal{O}_p)$ is the hypercohomology of this complex, i.e.

$$\mathrm{Ext}^k(\mathcal{O}_p,\mathcal{O}_p) \cong H^0(\bigwedge^k V \otimes \mathcal{O}_p) \cong \bigwedge^k V_p \tag{7}$$

Similarly, $\mathrm{Ext}^i(\mathcal{O}_p,\mathcal{E})$ can be computed by hypercohomology of

$$\mathcal{E} \to\xrightarrow{s} V \otimes \mathcal{E} \xrightarrow{s} \bigwedge^2 V \otimes \mathcal{E} \xrightarrow{s} \cdots \xrightarrow{s} \bigwedge^n V \otimes \mathcal{E} \tag{8}$$

which is the Koszul resolution of the skyscraper sheaf with stalk $\bigwedge^n V \otimes \mathcal{E}$ at p. This sequence is exact except in the last place, and the cokernel is a skyscraper sheaf with stalk $\bigwedge^n \otimes \mathcal{E}$ at p. Thus, $\mathrm{Ext}^n(\mathcal{O}_p,\mathcal{E}) \cong (\bigwedge^n V \otimes \mathcal{E})_p$ with all other groups zero. This is consistent with the Serre duality $\mathrm{Ext}^i(\mathcal{E},\mathcal{F}) \cong \mathrm{Ext}^{n-i}(\mathcal{F}, K_X \otimes \mathcal{E})^\vee$.

0.2. **Derived Categories.** The general idea is to work with complexes up to homotopy.

- Enlarging a category to include complexes makes it algebraically nicer (e.g. the derived category is *triangulated*) and less sensitive to the initial set of objects (we can restrict to a nice subcategory). For instance, for Fukaya categories, one can hope to allow objects like immersed Lagrangians implicitly.
- Even if we know how to define general objects, it is usually easier to replace them with complexes of nice objects. For instance, for $s \in H^0(\mathcal{L}), D = s^{-1}(0)$, we can exchange $\mathcal{O}_D$ with the complex $\{\mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X\}$.

Example. This makes it easier to perform intersection theory: for D_1, D_2 defined by sections s_1, s_2 of $\mathcal{L}_1, \mathcal{L}_2$, their homological intersection is

$$[D_1] \cdot [D_2] = c_1(\mathcal{L}_1) \cup c_1(\mathcal{L}_2) \cap [X] = c_1(\mathcal{L}_1|_{D_2}) \cdot [D_2] \tag{9}$$

If D_1 and D_2 intersect transversely, $\mathcal{O}_{D_1 \cap D_2} = \mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}$. We can also resolve this using the associated complex, i.e. apply $- \otimes \mathcal{O}_{D_2}$ to $\{\mathcal{L}_1^{-1} \xrightarrow{s_1} \mathcal{O}_X\}$, obtaining $\{\mathcal{L}_1^{-1}|_{D_2} \xrightarrow{s_1|_{D_2}} \mathcal{O}_{D_2}\}$. If $D_1 = D_2 = D$, $\mathcal{O}_D \otimes \mathcal{O}_D = \mathcal{O}_D$ is "too big" (because $\otimes$ is right exact but not exact). Using the associated complex still works, however, as we obtain $\{\mathcal{L}_1^{-1}|_D \xrightarrow{s|_D = 0} \mathcal{O}_D\}$ with kernel $\mathcal{L}^{-1}|_D$ and cokernel $\mathcal{O}_D$.

- When do we consider two complexes to be isomorphic? Having isomorphic cohomology is not enough. For instance, in algebraic topology, a theorem of Whitehead states that, for X, Y simply connected simplicial complexes, X and Y are homotopy equivalent $\Leftrightarrow$ $\exists\, Z$ and simplical maps $X \to Z, Y \to Z$ s.t. the chain maps $C^*(Z) \to C^*(X), C^*(Z) \to C^*(Y)$ are isomorphisms in cohomology.

Definition 1. *A chain map $f : C_* \to D_*$ (i.e. a collection of maps $f_i C_i \to D_i$ commuting with ∂) is a quasi-isomorphism if the induced maps on cohomology are isomorphisms.*

This is stronger than $H^*(C_*) \cong H^*(D_*)$.

Example. The complexes of $\mathbb{C}[x, y]$-modules $\mathbb{C}[x, y]^{\oplus 2} \to_{(x,y)} \mathbb{C}[x, y]$ and $\mathbb{C}[x, y] \to_0 \mathbb{C}$ have the same cohomology but are not quasi-isomorphic.